Continuous Random Variables (Cumulative Distributions) (From OCR 4768)

Q1, (Jan 2006, Q1i-iii)

Q1	$F(t) = 1 - e^{-t/3}$ $(t>0)$			
(i)	For median m, $\frac{1}{2} = 1 - e^{-m/3}$	M1		
	$\therefore e^{-m/3} = \frac{1}{2} \Rightarrow -\frac{m}{3} = \ln\frac{1}{2} = -0.6931$	M1	attempt to solve, here or for 90th percentile. Depends on previous M mark.	
	$\Rightarrow m = 2.079$	A1		
	For 90^{th} percentile p, $0.9 = 1 - e^{-p/3}$	M1		
	$\therefore e^{-p/3} = 0.1 \Rightarrow -\frac{p}{3} = \ln 0.1 = -2.3026$			
	$\Rightarrow p = 6.908$	A1		5
(ii)	$f(t) = \frac{d}{dt}F(t)$	M1		
	$= \frac{1}{3}e^{-t/3}$	A1	(for $t>0$, but condone absence of this)	
	$\mu = \int_0^\infty \frac{1}{3}te^{-t/3}dt$	M1	Quoting standard result gets 0/3 for the mean.	
	$= \frac{1}{3}\left\{\left[\frac{te^{-t/3}}{-1/3}\right]_0^\infty + 3\int_0^\infty e^{-t/3}dt\right\}$	M1	attempt to integrate by parts	
	$= [0-0] + \left[\frac{e^{-t/3}}{-1/3}\right]_0^\infty = 3$	A1		5
(iii)	$P(T>\mu) = $ [from cdf] $e^{-\mu/3} = e^{-1}$	M1	[or via pdf]	
	$=0.3679$	A1	ft c's mean (>0)	2

Q2, (Jun 2006, Q1i,ii)

	$f(x) = 12x^3 - 24x^2 + 12x, \qquad 0 \le x \le 1$			
(i)	$E(X) = \int_0^1 x f(x)\,dx$	M1	Integral for E(X) including limits (which may appear later).	
	$= 12\left[\frac{x^5}{5} - 2\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1$	A1	Successfully integrated.	
	$= 12\left[\frac{1}{5} - \frac{2}{4} + \frac{1}{3}\right] = 12 \times \frac{1}{30} = \frac{2}{5}$	A1	Correct use of limits leading to final answer. C.a.o.	
	For mode, $f'(x) = 0$	M1		
	$f'(x) = 12(3x^2 - 4x + 1) = 12(3x-1)(x-1)$ $\therefore f'(x) = 0$ for $x = 1$ and $x = \frac{1}{3}$	A1		
	Any convincing argument (e.g. $f''(x)$) that $\frac{1}{3}$ (and not 1) is the mode.	A1		6
(ii)	Cdf $F(x) = \int_0^x f(t)\,dt$ $= 12\left(\frac{x^4}{4} - 2\frac{x^3}{3} + \frac{x^2}{2}\right)$ $= 3x^4 - 8x^3 + 6x^2$	M1 A1	Definition of cdf, including limits (or use of "+c" and attempt to evaluate it), possibly implied later. Some valid method must be seen. Or equivalent expression; condone absence of domain [0,1].	
	$F\left(\frac{1}{4}\right) = \frac{3}{256} - \frac{8}{64} + \frac{6}{16} = \frac{3-32+96}{256} = \frac{67}{256}$ $F\left(\frac{1}{2}\right) = \frac{3}{16} - \frac{8}{8} + \frac{6}{4} = \frac{3-16+24}{16} = \frac{11}{16}$ $F\left(\frac{3}{4}\right) = \frac{3\times81}{256} - \frac{8\times27}{64} + \frac{6\times9}{16} = \frac{243}{256}$	B1	For all three; answers given; must show convincing working (such as common denominator)! Use of decimals is not acceptable.	3

Q3, (Jan 2007, Q1i-iv)

	$f(x) = k(1-x) \quad 0 \le x \le 1$			
(i)	$\int_0^1 k(1-x)\,dx = 1$ $\therefore k[x - \frac{1}{2}x^2]_0^1 = 1$ $\therefore k(1-\frac{1}{2}) - 0 = 1$ $\therefore k = 2$ Labelled sketch: straight line segment from (0,2) to (1,0).	M1 E1 G1 G1	Integral of f(x), including limits (possibly implied later), equated to 1. Convincingly shown. Beware printed answer. Correct shape. Intercepts labelled.	4
(ii)	$E(X) = \int_0^1 2x(1-x)\,dx$ $= [x^2 - \frac{2}{3}x^3]_0^1 = (1-\frac{2}{3}) - 0 = \frac{1}{3}$ $E(X^2) = \int_0^1 2x^2(1-x)\,dx$ $= [\frac{2}{3}x^3 - \frac{2}{4}x^4]_0^1 = (\frac{2}{3}-\frac{1}{2}) - 0 = \frac{1}{6}$ $Var(X) = \frac{1}{6} - (\frac{1}{3})^2$ $= \frac{1}{18}$	M1 A1 M1 M1 A1	Integral for E(X) including limits (which may appear later). Integral for E(X^2) including limits (which may appear later). Convincingly shown. Beware printed answer.	5
(iii)	$F(x) = \int_0^x 2(1-t)\,dt$ $= [2t - t^2]_0^x = (2x - x^2) - 0 = 2x - x^2$ $P(X > \mu) = P(X > \frac{1}{3}) = 1 - F(\frac{1}{3})$ $= 1 - (2 \times \frac{1}{3} - (\frac{1}{3})^2) = 1 - \frac{5}{9} = \frac{4}{9}$	M1 A1 M1 A1	Definition of cdf, including limits, possibly implied later. Some valid method must be seen. [for $0 \le x \le 1$; do not insist on this.] For 1 – c's F(μ). ft c's E(X) and F(x). If answer only seen in decimal expect 3 d.p. or better.	4
(iv)	$F\left(1-\frac{1}{\sqrt{2}}\right) = 2\left(1-\frac{1}{\sqrt{2}}\right) - \left(1-\frac{1}{\sqrt{2}}\right)^2$ $= 2 - \frac{2}{\sqrt{2}} - 1 + \frac{2}{\sqrt{2}} - \frac{1}{2} = \frac{1}{2}$ **Alternatively:** $2m - m^2 = \frac{1}{2}$ $\therefore m^2 - 2m + \frac{1}{2} = 0$ $\therefore m = 1 \pm \frac{1}{\sqrt{2}}$ so $m = 1 - \frac{1}{\sqrt{2}}$	M1 E1 M1 E1	Substitute $m = 1 - \frac{1}{\sqrt{2}}$ in c's cdf. Convincingly shown. Beware printed answer. Form a quadratic equation $F(m) = \frac{1}{2}$ and attempt to solve it. ft c's cdf provided it leads to a quadratic. Convincingly shown. Beware printed answer.	2

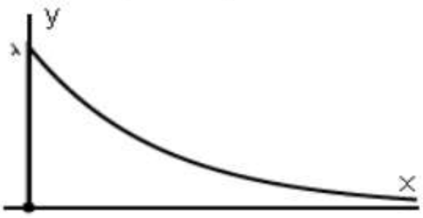
Q4, (Jun 2008, Q1a)

	$f(x) = k(20 - x) \qquad 0 \le x \le 20$			
(a) (i)	$\int_0^{20} k(20-x)\mathrm{d}x = \left[k\left(20x - \frac{x^2}{2}\right)\right]_0^{20} = k \times 200 = 1$	M1	Integral of f(x), including limits (which may appear later), set equal to 1. Accept a geometrical approach using the area of a triangle.	
	$\therefore k = \frac{1}{200}$	A1	C.a.o.	
	Straight line graph with negative gradient, in the first quadrant.	G1		
	Intercept correctly labelled (20, 0), with nothing extending beyond these points.	G1		
	Sarah is more likely to have only a short time to wait for the bus.	E1		5
(ii)	Cdf $\mathrm{F}(x) = \int_0^x \mathrm{f}(t)\mathrm{d}t$ $= \frac{1}{200}\left(20x - \frac{x^2}{2}\right)$	M1	Definition of cdf, including limits (or use of "+c" and attempt to evaluate it), possibly implied later. Some valid method must be seen.	
	$= \frac{x}{10} - \frac{x^2}{400}$	A1	Or equivalent expression; condone absence of domain [0, 20].	
	P(X > 10) = 1 – F(10)	M1	Correct use of c's cdf.	
	= 1 – (1 – ¼) = ¼	A1	f.t. c's cdf. Accept geometrical method, e.g area = ½(20 – 10)f(10), or similarity.	4
(iii)	Median time, m, is given by F(m) = ½. $\therefore \frac{m}{10} - \frac{m^2}{400} = \frac{1}{2}$	M1	Definition of median used, leading to the formation of a quadratic equation.	
	$\therefore m^2 - 40m + 200 = 0$	M1	Rearrange and attempt to solve the quadratic equation.	
	$\therefore m = 5.86$	A1	Other solution is 34.14; no explicit reference to/rejection of it is required.	3

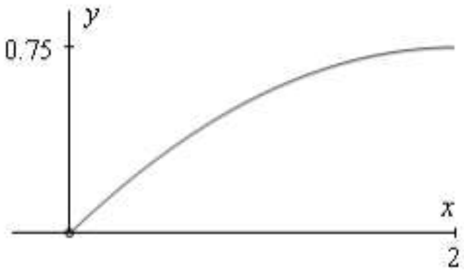
Q5, (Jun 2009, Q4i-iii)

	$f(x) = \frac{2x}{\lambda^2}$ for $0 < x < \lambda$, $\lambda > 0$			
(i)	f(x) > 0 for all x in the domain.	E1		
	$\int_0^\lambda \frac{2x}{\lambda^2}\,dx = \left[\frac{x^2}{\lambda^2}\right]_0^\lambda = \frac{\lambda^2}{\lambda^2} = 1$	M1	Correct integral with limits.	
		A1	Shown equal to 1.	3
(ii)	$\mu = \int_0^\lambda \frac{2x^2}{\lambda^2}\,dx = \left[\frac{2x^3/3}{\lambda^2}\right]_0^\lambda = \frac{2\lambda}{3}$	M1	Correct integral with limits.	
		A1	c.a.o.	
	$P(X < \mu) = \int_0^\mu \frac{2x}{\lambda^2}\,dx = \left[\frac{x^2}{\lambda^2}\right]_0^\mu$	M1	Correct integral with limits.	
	$= \frac{\mu^2}{\lambda^2} = \frac{4\lambda^2/9}{\lambda^2} = \frac{4}{9}$ which is independent of λ.	A1	Answer plus comment. ft c's μ provided the answer does not involve λ.	4
(iii)	Given $E(X^2) = \frac{\lambda^2}{2}$			
	$\sigma^2 = \frac{\lambda^2}{2} - \frac{4\lambda^2}{9} = \frac{\lambda^2}{18}$	M1	Use of Var(X) = E(X^2) – E(X)^2.	
		A1	c.a.o.	2

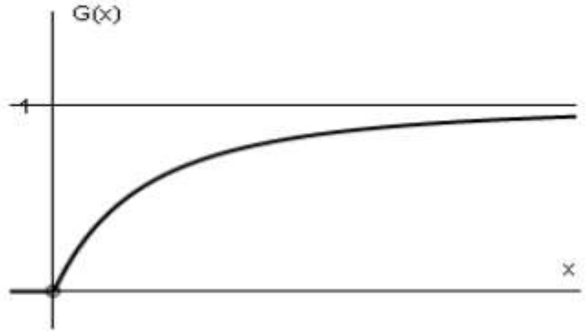
Q6, (Jun 2010, Q4i,ii)

	$f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$, where $\lambda > 0$.	Given $\int_0^\infty x^r e^{-\lambda x} dx = \frac{r!}{\lambda^{r+1}}$	
(i)	$\int_0^\infty f(x) dx = \int_0^\infty \lambda e^{-\lambda x} dx$ $= \left[-e^{-\lambda x}\right]_0^\infty$ $= \left(0 - (-e^0)\right) = 1$	M1 Integration of $f(x)$. M1 Use of limits or the given result. A1 Convincingly obtained (Answer given.)	
		G1 Curve, with negative gradient, in the first quadrant only. Must intersect the y-axis. G1 $(0, \lambda)$ labelled; asymptotic to x-axis.	[5]
(ii)	$E(X) = \int_0^\infty \lambda x e^{-\lambda x} dx$ $= \lambda \frac{1}{\lambda^2} = \frac{1}{\lambda}$ $E(X^2) = \int_0^\infty \lambda x^2 e^{-\lambda x} dx$ $= \lambda \frac{2}{\lambda^3} = \frac{2}{\lambda^2}$ $\text{Var}(X) = E(X^2) - E(X)^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$	M1 Correct integral. A1 c.a.o. (using given result) M1 Correct integral. A1 c.a.o. (using given result) M1 Use of $E(X^2) - E(X)^2$ A1	[6]

Q7, (Jan 2013, Q2i,ii)

(i)		G1 G1 G1 [3]	Curve with positive gradient, through the origin and in the first quadrant only. Correct shape for an inverted parabola ending at maximum point. End point (2, 3/4) labelled.
(ii)	$\mathrm{E}(X)=\frac{3}{16}\int_0^2(4x^2-x^3)\,\mathrm{d}x$	M1	Correct integral for E(X) with limits (which may appear later).
	$=\frac{3}{16}\left[\frac{4x^3}{3}-\frac{x^4}{4}\right]_0^2$	M1	Correctly integrated. Dep on previous M1.
	$=\frac{3}{16}\left\{\left(\frac{32}{3}-\frac{16}{4}\right)-0\right\}$		
	$=\frac{5}{4}$	A1	Limits used correctly to obtain PRINTED ANSWER (BEWARE) convincingly. Condone absence of "–0".
	$\mathrm{E}(X^2)=\frac{3}{16}\int_0^2(4x^3-x^4)\,\mathrm{d}x$	M1	Correct integral for E(X) with limits (which may appear later).
	$=\frac{3}{16}\left[x^4-\frac{x^5}{5}\right]_0^2$	M1	Correctly integrated. Dep on previous M1.
	$=\frac{3}{16}\left\{\left(16-\frac{32}{5}\right)-0\right\}$		
	$=\frac{9}{5}$	A1	Limits used correctly to obtain result. Condone absence of "–0".
	$\mathrm{Var}(X)=\frac{9}{5}-\left(\frac{5}{4}\right)^2=\frac{19}{80}$	M1	Use of $\mathrm{Var}(X)=\mathrm{E}(X^2)-\mathrm{E}(X)^2$.
	$\mathrm{sd}=\sqrt{\frac{19}{80}}=0.487(3)$	A1 [8]	cao

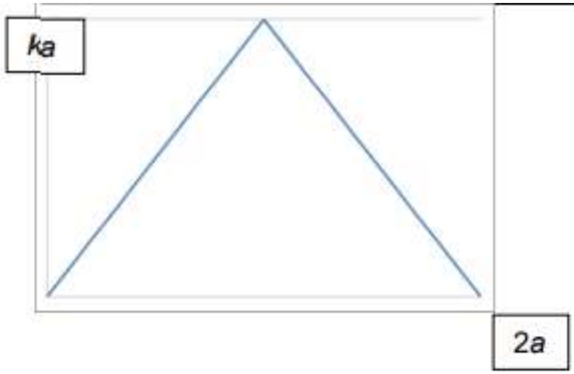
Q8, (Jun 2011, Q3)

(i) (A)		M1 A1 A1	Increasing curve, through (0, 0), in first quadrant only. Asymptotic behaviour. Asymptote labelled; condone absence of axis labels.	3
(B)	For the UQ $G(u) = 0.75$ $\therefore \left(1+\frac{u}{200}\right)^{-2} = \frac{1}{4} \quad \therefore u = 200$ For the LQ $G(l) = 0.25$ $\therefore \left(1+\frac{l}{200}\right)^{-2} = \frac{3}{4} \quad \therefore l = 200\left(\frac{2}{\sqrt{3}}-1\right) = 30.94...$ $\therefore \text{IQR} = 200 - 30.94 = 169(.06)$ For an outlier $x >$ UQ $+1.5 \times$ IQR $= 200 + 1.5 \times 169$ $=453(.58) \approx 454$ (nearest hour)	M1 A1 A1 M1 M1 E1	Use of G(x) for either quartile. c.a.o. c.a.o. UQ – LQ UQ +1.5 × IQR. Answer given; must be obtained genuinely.	6
(ii) (A)	$F(x) = \int_0^x \frac{1}{200} e^{\frac{-t}{200}} dt$ $= \left[-e^{\frac{-t}{200}} \right]_0^x = \left(-e^{\frac{-x}{200}}\right) - \left(-e^{\frac{-0}{200}}\right) = 1 - e^{\frac{-x}{200}}$	M1 A1 E1	Correct integral, including limits (which may be implied subsequently). Correctly integrated. Limits used. Answer given; must be shown convincingly. Condone the omission of x < 0 part. Allow use of "+ c" with F(0) = 0.	3
(B)	$P(X > 50) = 1 - F(50)$ $= e^{\frac{-50}{200}} = e^{-0.25}$	M1 E1	Use of 1 – F(x) Answer given: must be convincing. (= 0.7788(0))	2
(C)	$P(X > 400) = e^{\frac{-400}{200}} = 0.1353(35)$ $P(X > 450) = e^{\frac{-450}{200}} = 0.1053(99)$ $P(X > 450 \mid X > 400) = \frac{P(X > 450)}{P(X > 400)}$ $= \frac{e^{\frac{-450}{200}}}{e^{\frac{-400}{200}}} = e^{\frac{-50}{200}} = e^{-0.25} \ (= 0.7788)$	B1 B1 M1 A1	Accept any form. Accept any form. Conditional probability. Not $P(X > 50) \times P(X > 400)$ unless <u>clearly</u> justified. Accept division of decimals, 3dp or better. Accept a.w.r.t. 0.778 or 0.779.	4
				18

Q9, (Jun 2012, Q4ii-v)

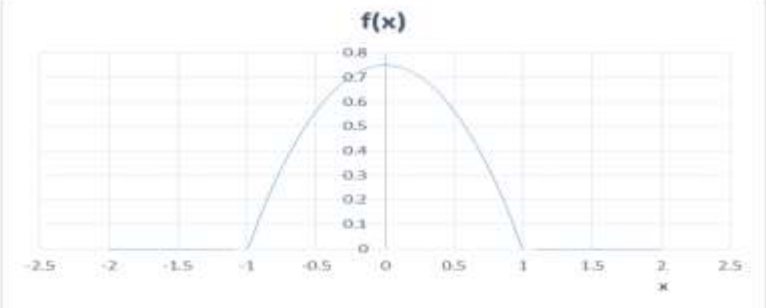
(ii)		Mean = 5/3 $\therefore\ \lambda = 0.6$	B1 [1]	
(iii)		$F(t) = \int_0^t 0.6e^{-0.6x}\,dx$ $= \left[-e^{-0.6x}\right]_0^t$ $= \left(-e^{-0.6t} - (-e^0)\right) = 1 - e^{-0.6t}$	M1 A1 A1 [3]	Correct integral with limits (which may be implied subsequently). Allow use of "+ c" accompanied by a valid attempt to evaluate it. Correctly integrated. Limits used or c evaluated correctly. Accept unsimplified form. If final answer is given in terms of λ then allow max M1A1A0.
(iv)		$P(T > 1) = 1 - F(1)$ $= 1 - \left(1 - e^{-0.6}\right) = 0.5488$	M1 A1 [2]	ft c's F(t). cao Allow any exact form of the correct answer.
(v)		$F(m) = \frac{1}{2} \quad \therefore 1 - e^{-0.6m} = \frac{1}{2}$ $\therefore e^{-0.6m} = \frac{1}{2} \quad \therefore -0.6m = -\ln 2 \quad \therefore m = \frac{\ln 2}{0.6}$ $m = 1.155$ (days)	M1 M1 A1 [3]	Use of definition of median. Allow use of c's F(t). Convincing attempt to rearrange to "m = …", to include use of logs. Cao obtained only from the correct F(t). Must be evaluated. Require 2 to 4 sf; condone 5.

Q10, (Jun 2014, Q4i-iii)

(i)		 $E(X) = a$ because the distribution is symmetrical about $x = a$	G1 G1 B1 **[3]**	Shape Scales on axes Do not allow integration method
(ii)		Total area = $\frac{1}{2}2a.ka$ Or $\int_0^a kx\,dx + \int_a^{2a} k(2a - x)\,dx$ $k\left[\frac{x^2}{2}\right]_0^a + k\left[2ax - \frac{x^2}{2}\right]_a^{2a}$ $k\left(\frac{a^2}{2} - 0\right) + k\left(2a^2 - \frac{3a^2}{2}\right)$ $= ka^2$ $ka^2 = 1$ $k = \frac{1}{a^2}$	M1 A1 A1 **[3]**	Attempting to find area of triangle or setting up the correct integrals including limits (which may appear later). oe Correctly finding area in terms of k, a Equating area to 1 and convincingly obtaining result. Answer given

(iii)	$\text{Var}(X) = k\int_0^a x^3\,dx + k\int_a^{2a} (2ax^2 - x^3)\,dx - a^2$	M1	Correct integral for $E(X^2)$ including limits (which may appear later).
	$k\left[\frac{x^4}{4}\right]_0^a + k\left[\frac{2ax^3}{3} - \frac{x^4}{4}\right]_a^{2a} - a^2$	M1	Correctly integrated (dependent on M1 above)
	$\frac{a^2}{4} + \frac{16a^2}{3} - 4a^2 - \frac{2a^2}{3} + \frac{a^2}{4} - a^2$	M1	Using $E(X^2) - (E(X))^2$
	$\text{Var}(X) = \frac{a^2}{6}$	A1	cao
		[4]	

Q11, (Jun 2016, Q3i-iv)

i	$k\int_{-1}^{1}(1-x^2)\mathrm{d}x = 1 \;\left(\to k\left[x - \frac{x^3}{3}\right]_{-1}^{1} = 1\right)$	M1	Correct integral including limits
	$\to \frac{4k}{3} = 1$	M1	(const)×k = 1
	$\to k = \frac{3}{4}$	A1	cao
		[3]	
ii		B1 B1 B1	General shape between -1 and +1 Axes labelled with scales and intercepts (FT their k) Nothing outside $\lvert x\rvert < 1$
		[3]	
iii	$E(X) = 0 \to V(X) = E(X^2)$	B1	for $E(X) = 0$
	$V(X) = \frac{3}{4}\int_{-1}^{1}(x^2 - x^4)\mathrm{d}x = \frac{3}{4}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1}$	M1	for correct integral including limits
	$= \frac{1}{5}$	A1	cao (ignore mistakes in working)
		[3]	

iv	$\frac{3}{4}\int_0^q (1 - x^2)\mathrm{d}x = \frac{1}{4}$	M1	Correct limits and equality
	integration $= \frac{3}{4}\left[x - \frac{x^3}{3}\right]$	B1	f/t their *k*
	$\rightarrow q - \frac{q^3}{3} = \frac{1}{3}$ or $\rightarrow q^3 - 3q + 1 = 0$	A1	any correct simplified (3-term) cubic
	$g(0.345) = 0.006$ $g(0.355) = -0.02$	M1	(allow correct alternative) If solving using calculator: state all three solutions
	Change of sign $\rightarrow 0.345 < q < 0.355$ So upper quartile = 0.35 to 2 dp	E1 [5]	must be explained clearly If solving by calculator: explain why only one works