Q1. (Cambridge 9795/01, 2011 Specimen, Q13)

Let \( I_n = \int_1^e (\ln x)^n \, dx \), where \( n \) is a positive integer.

(i) By considering \( \frac{d}{dx}(x(\ln x)^n) \), or otherwise, show that \( I_n = e - nI_{n-1} \). \[4\]

(ii) Let \( J_n = \frac{I_n}{n!} \). Prove by induction that

\[
\sum_{r=2}^{n} \frac{(-1)^r}{r!} = \frac{1}{e} \left( 1 + (-1)^n J_n \right)
\]

for all positive integers \( n \geq 2 \). \[10\]

Q2. (Cambridge 9795/01, 2013, Q12)

Given \( y = xe^{2x} \),

(i) find the first four derivatives of \( y \) with respect to \( x \), \[4\]

(ii) conjecture an expression for \( \frac{d^n y}{dx^n} \) in the form \((ax + b)e^{2x}\), where \( a \) and \( b \) are functions of \( n \), \[2\]

(iii) prove by induction that your result holds for all positive integers \( n \). \[5\]

Q3. (Cambridge 9795/01, 2016, Q11)

(i) The sequence of Fibonacci Numbers \( \{F_n\} \) is given by

\[ F_1 = 1, \quad F_2 = 1 \quad \text{and} \quad F_{n+2} = F_n + F_{n+1} \quad \text{for } n \geq 2. \]

Write down the values of \( F_3 \) to \( F_6 \). \[1\]

(ii) The sequence of functions \( \{p_n(x)\} \) is given by

\[ p_1(x) = x + 1 \quad \text{and} \quad p_{n+1}(x) = 1 + \frac{1}{p_n(x)} \quad \text{for } n \geq 1. \]

(a) Find \( p_2(x) \) and \( p_3(x) \), giving each answer as a single algebraic fraction, and show that \( p_4(x) = \frac{3x + 5}{2x + 3} \). \[3\]

(b) Conjecture an expression for \( p_n(x) \) as a single algebraic fraction involving Fibonacci numbers, and prove it by induction for all integers \( n \geq 2 \). \[5\]
Q4. (Cambridge 9795/01, 2018, Q8)

(i) Write down the values of the constants $a$ and $b$ for which $m^5 = \frac{1}{6} m^3 (am^2 + 2) - \frac{1}{12} m^2 (bm)$. \[1\]

(ii) Prove by induction that $\sum_{r=1}^{n} r^5 = \frac{1}{6} n^3 (n + 1)^3 - \frac{1}{12} n^2 (n + 1)^2$ for all positive integers $n$. \[7\]

Q5. (Cambridge 9795/01, 2017, Q12)

For each positive integer $n$, the function $F_n$ is defined for all real angles $\theta$ by

$$F_n(\theta) = c^{2n} + s^{2n}$$

where $c = \cos \theta$ and $s = \sin \theta$.

(i) Prove the identity

$$F_{n+2}(\theta) - \frac{1}{4} \sin^2 2\theta \times F_{n+1}(\theta) = F_{n+3}(\theta).$$

Let $z$ denote the complex number $c + is$.

(ii) Using de Moivré’s theorem,

(a) express $z + z^{-1}$ and $z - z^{-1}$ in terms of $c$ and $s$ respectively.

(b) prove the identity $8(c^6 + s^6) = 3 \cos 4\theta + 5$ and deduce that

$$c^6 + s^6 = \cos^2 2\theta + \frac{1}{4} \sin^2 2\theta.$$ \[7\]

(iii) Prove by induction that, for all positive integers $n$,

$$c^{2n+4} + s^{2n+4} \leq \cos^2 2\theta + \frac{1}{2^{n+1}} \sin^2 2\theta.$$

[You are given that the range of the function $F_n$ is $\frac{1}{2^n} \leq F_n(\theta) \leq 1.$] \[7\]