Proof By Induction (Challenging) Exam Questions MS

Q1, (Cambridge 9795/01, 2011 Specimen, Q13)

(i) Differentiate $x(\ln x)^n$ as a product	M1	
Obtain $(\ln x)^n + n(\ln x)^{n-1}$	A1	
Deduce $\int_1^e (\ln x)^n \, dx + n\int_1^e (\ln x)^{n-1} \, dx = \left[x(\ln x)^n\right]_1^e$	M1	
Obtain given result correctly	A1	4
(ii) $I_1 = \int_1^e \ln x \, dx = \left[x\ln x - x\right]_1^e = 1$ (e.g. by parts)	B1	
$J_n = \frac{e}{n!} - J_{n-1}$ noted at any point	B1	
$J_1 = 1 \Rightarrow J_2 = \frac{1}{2}e - 1$	B1	
$\sum_{r=2}^{2} \frac{(-1)^r}{r!} = \frac{1}{2!} = \frac{1}{2}$ and $\frac{1}{e}(1+(-1)^2 J_2) = \frac{1}{e}(1+\frac{1}{2}e-1) = \frac{1}{2}$ so result is true for $n = 2$	B1	
Assume $\sum_{r=2}^{k} \frac{(-1)^r}{r!} = \frac{1}{e}(1+(-1)^k J_k)$ induction hypothesis	B1	
Then $\sum_{r=2}^{k+1} \frac{(-1)^r}{r!} = \sum_{r=2}^{k} \frac{(-1)^r}{r!} + \frac{(-1)^{k+1}}{(k+1)!}$ attempt at $S_k + u_{k+1}$	M1	
Then $\sum_{r=2}^{k+1} \frac{(-1)^r}{r!} = \frac{1}{e}(1+(-1)^k J_k) + \frac{(-1)^{k+1}}{(k+1)!}$ use of induction hypothesis	M1	
$J_k = \frac{e}{(k+1)!} - J_{k+1}$ used to express this in terms of J_{k+1}	M1	
$\sum_{r=2}^{k+1} \frac{(-1)^r}{r!} = \frac{1}{e}\left(1+(-1)^k\left[\frac{e}{(k+1)!} - J_{k+1}\right]\right) + \frac{(-1)^{k+1}}{(k+1)!}$ legitimately rearranged into		
form $\frac{1}{e}(1+(-1)^{k+1}J_{k+1})$	A1	
Induction explanation rounded off convincingly	B1	10

Q2, (Cambridge 9795/01, 2013, Q12)

(i)	$y'(x) = (2x+1)\,e^{2x}$, $\quad y''(x) = (4x+4)\,e^{2x}$, $y'''(x) = (8x+12)\,e^{2x}$, $\quad y^{(4)}(x) = (16x+32)\,e^{2x}$	B1 B1 B1 B1	[4]
(ii)	Conjecture $\frac{d^n y}{dx^n} = (2^n x + n.2^{n-1})\,e^{2x}$ One mark each: coefft. of x, constant	B1 B1	[2]
(iii)	Diffferentiating their conjectured expression (must be linear × e^{2x})	M1	
	$\frac{d^{n+1}y}{dx^{n+1}} = 2 \times (2^n x + n.2^{n-1})\,e^{2x} + 2^n \times e^{2x}$ **FT** max 1/2	A1 A1	
	$= (2^{n+1}x + (n+1).2^{(n+1)-1})\,e^{2x}$ Shown of correct form	A1	
	Usual induction round-up/explanation of proof, including clear demonstration that $(n+1)^{th}$ formula is in the right form.	E1	[5]

Q3, (Cambridge 9795/01, 2016, Q11)

(i)	$F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ F_6 = 8$	B1 [1]	all
(ii) (a)	$p_2(x) = 1 + \frac{1}{x+1} = \frac{x+2}{x+1}$	B1	
	$p_3(x) = \frac{2x+3}{x+2}$	B1	
	$p_4(x) = \frac{3x+5}{2x+3}$	B1 [3]	(AG))
(b)	$p_n(x) = \frac{F_n\,x + F_{n+1}}{F_{n-1}\,x + F_n}$	B1	
	Result is true for $n = 2$ (and 3 and 4)	B1	May be mentioned in later in their "round up"
	Assuming $p_k(x) = \frac{F_k\,x + F_{k+1}}{F_{k-1}\,x + F_k}$ (not separate from their conjecture)		
	$p_{k+1}(x) = 1 + \frac{F_{k-1}\,x + F_k}{F_k\,x + F_{k+1}}$	M1	
	$= \frac{F_k\,x + F_{k+1}}{F_k\,x + F_{k+1}} + \frac{F_{k-1}\,x + F_k}{F_k\,x + F_{k+1}}$		
	$= \frac{(F_k + F_{k-1})\,x + (F_k + F_{k+1})}{F_k\,x + F_{k+1}}$	M1	Collecting coeffts. into successive Fib. terms
	$= \frac{F_{k+1}\,x + F_{k+2}}{F_k\,x + F_{k+1}}$	A1	
	which is the required formula with $n = k + 1$. Accept this as sufficient that proof follows by induction.	[5]	

Q4, (Cambridge 9795/01, 2018, Q8)

8(i)	$a = 6,\ b = 4$	B1	
8(ii)	For $n = 1$, LHS $= 1^5 = 1$ and RHS $= \frac{1}{6}.2^3 - \frac{1}{12}.2^2 = \frac{4}{3} - \frac{1}{3} = 1$ so that the result is true for $n = 1$	B1	Both sides must be established
	Assume that $\sum_{r=1}^{k} r^5 = \frac{1}{6}k^3(k+1)^3 - \frac{1}{12}k^2(k+1)^2$	M1	Induction hypothesis clearly stated somewhere
	Then $\sum_{r=1}^{k+1} r^5 = \frac{1}{6}k^3(k+1)^3 - \frac{1}{12}k^2(k+1)^2 + (k+1)^5$	M1	Attempt at S_{k+1} with S_k used
	$= \frac{1}{6}k^3(k+1)^3 - \frac{1}{12}k^2(k+1)^2 + \frac{1}{6}(k+1)^3\left[6(k+1)^2 + 2\right] - \frac{1}{12}(k+1)^2\left[4(k+1)\right]$	M1	Use of (i)'s result with $m = k + 1$ for the $(k + 1)^5$ term
	$= \frac{1}{6}(k+1)^3\left[k^3 + 6(k^2 + 2k + 1) + 2\right] - \frac{1}{12}(k+1)^2\left[k^2 + 4(k+1)\right]$	M1	Terms collected appropriately
	$= \frac{1}{6}(k+1)^3(k+2)^3 - \frac{1}{12}(k+1)^2(k+2)^2$	A1	Legitimately shown so
	Hence result true for $n = k \Rightarrow$ result true for $n = k + 1$. Since result true for $n = 1$, it follows that it is true for $n = 2$, $n = 3$, etc. and the result is true for all positive integers n by induction	E1	Induction process clearly explained: minimum requirement is (P_1 ✓) and (P_k ✓ $\Rightarrow P_{k+1}$ ✓)

(ii)	**Alt. I** For $n = 1$, LHS = $1^5 = 1$ and RHS = $\frac{1}{6}.2^3 - \frac{1}{12}.2^2 = \frac{4}{3} - \frac{1}{3} = 1$ so that the result is true for $n = 1$	**B1**	Both sides must be established
	Assume that $\sum_{r=1}^{k} r^5 = \frac{1}{6}k^3(k+1)^3 - \frac{1}{12}k^2(k+1)^2$	**M1**	Induction hypothesis clearly stated somewhere
	Then $\sum_{r=1}^{k+1} r^5 = \frac{1}{6}k^3(k+1)^3 - \frac{1}{12}k^2(k+1)^2 + (k+1)^5$	**M1**	Attempt at S_{k+1} with S_k used
	$= \frac{1}{12}(k+1)^2(2k^4 + 14k^3 + 35k^2 + 36k + 12)$	**M1**	Factorising out the $(k+1)^2$
	$= \frac{1}{12}(k+1)^2(k^2 + 4k + 4)(2k^2 + 6k + 3)$		
	$= \frac{1}{12}(k+1)^2(k+2)^2(2(k+1)(k+2) - 1)$	**M1**	Factorising and splitting the final factor suitably
	$= \frac{1}{6}(k+1)^3(k+2)^3 - \frac{1}{12}(k+1)^2(k+2)^2$	**A1**	Legitimately shown so
	Hence result true for $n = k \Rightarrow$ result true for $n = k + 1$. Since result true for $n = 1$, it follows that it is true for $n = 2$, $n = 3$, etc. and the result is true for all positive integers n by induction	**E1**	Induction process clearly explained: minimum requirement is (P_1 ✓) and (P_k ✓ $\Rightarrow P_{k+1}$ ✓)

Q5, (Cambridge 9795/01, 2017, Q12)

(i)	**Method I** $F_{n+2}(\theta) - \frac{1}{4}\sin^2(2\theta)\, F_{n+1}(\theta)$ $\equiv (c^2 + s^2)(c^{2n+4} + s^{2n+4})$ $- \frac{1}{4}(2sc)^2(c^{2n+2} + s^{2n+2})$	**M2**	**M1** all F_n terms **M1** $\sin 2\theta$ form
	$\equiv c^{2n+6} + c^2 s^{2n+4} + s^2 c^{2n+4} + s^{2n+6}$ $- c^2 s^2 (c^{2n+2} + s^{2n+2})$	**A1**	
	$\equiv c^{2n+6} + s^{2n+6} \equiv F_{n+3}(\theta)$	**A1**	**AG**
	Method II $\equiv c^{2n+4} + s^{2n+4} - s^2 c^2 (c^{2n+2} + s^{2n+2})$	**M1**	Use of $\sin 2\theta$ form
	$\equiv c^{2n+4} + s^{2n+4} - s^2 c^{2n+4} - c^2 s^{2n+4}$	**A1**	
	$\equiv (1 - s^2)c^{2n+4} + (1 - c^2)s^{2n+4}$	**M1**	
	$\equiv c^{2n+6} + s^{2n+6} \equiv F_{n+3}(\theta)$	**A1**	**AG**

(ii)(a)	Use of $z = c + is$ and $z^{-1} = c - is$	M1	
	$z + z^{-1} = 2c$ and $z - z^{-1} = 2is$	A2	A1 for each
(ii)(b)	**Method I** $(2c)^6 = (z + z^{-1})^6 = z^6 + 6z^4 + 15z^2 + 20$ $+ 15z^{-2} + 6z^{-4} + z^{-6}$	M1	
	$= 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$	A1	
	$-(2s)^6 = (z - z^{-1})^6 = z^6 - 6z^4 + 15z^2 - 20$ $+ 15z^{-2} - 6z^{-4} + z^{-6}$ $= 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$	B1	**FT** (Must have – sign)
	Subtracting: $64(c^6 + s^6) = 12(z^4 + z^{-4}) + 40$ $= 12 \,.\, 2\cos 4\theta + 40$	M1	
	Dividing by 8: $8(c^6 + s^6) = 3\cos 4\theta + 5$	A1	**AG**
	Use of $\cos 4\theta = 2\cos^2 2\theta - 1$ and $1 = \cos^2 2\theta + \sin^2 2\theta$	M1	
	$\Rightarrow c^6 + s^6 = \frac{3}{8}(2\cos^2 2\theta) + (-\frac{3}{8} + \frac{5}{8})(\cos^2 2\theta + \sin^2 2\theta)$ $= \cos^2 2\theta + \frac{1}{4}\sin^2 2\theta$	A1	**AG**
(ii)(b)	**Method II** $\cos 4\theta = \text{Re}(c + is)^4$	M1	
	$= c^4 - 6c^2s^2 + s^4 = c^4 - 6c^2(1 - c^2) + (1 - c^2)^2$ $= 8c^4 - 8c^2 + 1$	A1	
	$c^6 + s^6 = c^6 + (1 - c^2)^3 = c^6 + 1 - 3c^2 + 3c^4 - c^6$	M1	
	$= 3c^4 - 3c^2 + 1$	A1	
	so that $8(c^6 + s^6) = 3\cos 4\theta + 5$	A1	**AG**
	Use of $\cos 4\theta = \cos^2 2\theta - \sin^2 2\theta$ and $1 = \cos^2 2\theta + \sin^2 2\theta$	M1	
	$\Rightarrow 8(c^6 + s^6) = 3\cos 4\theta + 5$ $= 3(\cos^2 2\theta - \sin^2 2\theta) + 5(\cos^2 2\theta + \sin^2 2\theta)$ $\Rightarrow c^6 + s^6 = \cos^2 2\theta + \frac{1}{4}\sin^2 2\theta$	A1	**AG**

(iii)	Case for $n = 1$ established in (ii) (b):	**B1**	noted explicitly (possibly at end)
	Assume $c^{2k+4} + s^{2k+4} \leqslant \cos^2 2\theta + \frac{1}{2^{k+1}}\sin^2 2\theta$	**B1**	i.e. the case for $n = k$
	A clear statement of the result must be given, possibly within what follows Then $c^{2k+6} + s^{2k+6} =$ $c^{2k+4} + s^{2k+4} - \frac{1}{4}\sin^2 2\theta\left(c^{2k+2} + s^{2k+2}\right)$	**M1**	attempt at $n = k + 1$ case using **(i)**'s identity
	$\leqslant \cos^2 2\theta + \frac{1}{2^{k+1}}\sin^2 2\theta - \frac{1}{4}\sin^2 2\theta\left(c^{2k+2} + s^{2k+2}\right)$	**M1**	use of the induction hypothesis (i.e. the $n = k$ case)
	$= \cos^2 2\theta + \frac{1}{2^{k+2}}\sin^2 2\theta - \frac{1}{4}\sin^2 2\theta\left(c^{2k+2} + s^{2k+2} - \frac{1}{2^k}\right)$	**M1A1**	splitting up the $\sin^2 2\theta$ term into two equal parts
	$\leqslant \cos^2 2\theta + \frac{1}{2^{k+2}}\sin^2 2\theta$ Proof follows by induction since $\sin^2 2\theta \geqslant 0$ and given result that $c^{2k+2} + s^{2k+2} \geqslant \frac{1}{2^k}$	**A1**	