Maclaurin Expansion Exam Questions MS (From OCR 4726)

Q1, (Jan 2006, Q1)

(i) Use standard \( \ln(1+3x) = 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} \)

\[ = 3x - 9x^2/2 + 9x^3 \]

M1 Allow e.g. \( 3x^2, 2! \) etc.
M1 Attempt to simplify \( (3x)^2 \) etc.
A1 cao

(ii) Produce \( (1 + x + x^2/2) \)

Get \( 3x - 3x^2/2 + 6x^3 \)

B1
M1 Mult. 2 reasonable attempts, each of 3 terms (non-zero)
A1 √ From their series
SC M1 Reasonable attempt at diff. and replace \( x = 0 \) (2 correct)
M1 √ Put their values into correct Maclaurin expansion
A1 cao
(Applies to either/both parts)

Q2, (Jan 2007, Q1)

(i) \( f(0) = \ln 3 \)

\[ f'(0) = \frac{1}{3} \]

\[ f''(0) = -\frac{1}{9} A. G. \]

Bl
Bl
B1 Clearly derived

(ii) Reasonable attempt at Maclaurin

\[ f(x) = \ln 3 + \frac{1}{3}x + \frac{1}{18}x^2 \]

MI Form \( \ln 3 + ax + bx^2 \), with \( a, b \) related to \( f '' \)
A1/√ On their values off’and \( f '' \)
SR Use \( \ln(3+x) = \ln 3 + \ln(1 + \frac{1}{3} x) \)
M1 Use Formulae Book to get

\[ \ln 3 + \frac{y_3x - y_2(1+y_2)}{2} = \ln 3 + \frac{y_3x - 1/\ln y_2}{2} \]
Q3, (Jan 2009, Q1)

(i) Give $1 + 2x + (2x)^2/2$
    Get $1 + 2x + 2x^2$

M1 Reasonable 3 term attempt e.g. allow $2x^2/2$
A1 cao

(ii) $\ln(1+2x+2x^2) + (1-2x+2x^2)$
    $\ln(2+4x^2) = \ln2 + \ln(1 + 2x^2)$
    $\ln2 + 2x^2$

M1 Attempt to sub for $e^{2x}$ and $e^{-2x}$
A1 $\sqrt{\text{On their part (i)}}$
M1 Use of log law in reasonable expression
A1 cao

SC Use of Maclaurin for $f''(x)$ and $f''(x)$
M1 One correct
A1 Attempt $f(0), f'(0)$ and $f''(0)$
M1 Get cao
A1

Q4, (Jan 2010, Q2)

(i) Find $f'(x) = 1/(1 + (1 + x)^2)$
    Get $f(0) = \frac{1}{4}\pi$ and $f'(0) = \frac{1}{2}$
    Attempt $f''(x)$

M1 Quoted or derived; may be simplified or
    left as $\sec^2y\, dy/dx = 1$
A1 $\sqrt{\text{On their } f'(0); \text{allow } f(0)=0.785 \text{ but not } 45}$
M1 Reasonable attempt at chain/quotient rule
    or implicit differentiation
A1 A.G.

(ii) Attempt Maclaurin as $af(0) + bf'(0) + cf''(0)$
    Get $\frac{1}{4}\pi + \frac{1}{2}x - \frac{1}{4}x^2$

M1 Using their $f(0)$ and $f'(0)$
A1 Cao; allow 0.785
<table>
<thead>
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<th>Question</th>
<th>Solution</th>
<th>Marks</th>
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<tbody>
<tr>
<td>(i)</td>
<td>( f'(x) = -\sin x \cdot e^{-x} + \cos x \cdot e^{-x} ) ( \Rightarrow f'(0) = 1 )</td>
<td>M1, A1, A1</td>
<td>Diffn using product correctly. For both values www</td>
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<td></td>
<td>( f(0) = 0 )</td>
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<td>(ii)</td>
<td>( f'(x) = \cos x \cdot e^{-x} - \sin x \cdot e^{-x} = \cos x \cdot e^x - f(x) )</td>
<td>M1</td>
<td>Diffn</td>
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<tr>
<td></td>
<td>( f''(x) = -f'(x) - \cos x \cdot e^{-x} - f(x) )</td>
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<td></td>
<td>( = -f'(x) - f'(x) - f(x) - f(x) )</td>
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<td></td>
<td>( f''(x) = -2f'(x) - 2f(x) ) OR (-2 \cos x \cdot e^{-x} ) Showing the two equal</td>
<td>A1, A1, A1</td>
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<td>( f''(0) = -2 )</td>
<td></td>
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<tr>
<td>(iii)</td>
<td>( f''(x) = -2f'(x) - 2f(x) ) ( \Rightarrow f''(x) = -2f''(x) - 2f'(x) ) oe</td>
<td>B1, B1</td>
<td>Not involving trig or exp fns ( = -f'' + 2f ) Or ( 2f'' + 4f )</td>
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<td></td>
<td>( \Rightarrow f''(0) = 4 - 2 = 2 )</td>
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<td>(iv)</td>
<td>( f(x) = x - x^2 + \frac{x^3}{3} )</td>
<td>M1, A1</td>
<td></td>
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<tr>
<td>Alternative:</td>
<td>Write down correct series expansion for ( e^x ) and ( \sin x ) and multiply</td>
<td>M1, A1</td>
<td></td>
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</tbody>
</table>
\[ \frac{dy}{dx} = \frac{1}{1 - \left(\frac{1-x}{3+x}\right)^2} \times \frac{-(3+x)-(1-x)}{(3+x)^2} \]

\[ \Rightarrow \frac{dy}{dx} = \left(\frac{-4}{(3+x)^2-(1-x)^2}\right) = \frac{k}{1+x} \]

\[ \Rightarrow \frac{dy}{dx} = \frac{-1}{2(1+x)} \]

\[ \Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{2(1+x)^2} \]

When \( x = 0 \), \( y = \tanh^{-1} \frac{1}{3} \) or \( \frac{1}{2} \ln 2 \) or \( \ln \sqrt{2} \)

\[ \frac{dy}{dx} = -\frac{1}{2} \]

\[ \frac{d^2y}{dx^2} = \frac{1}{2} \]

\[ \Rightarrow y = \tanh^{-1} \frac{1}{3} + \left( -\frac{1}{2} \right) x + \left( \frac{1}{2} \right) x^2 \]

\[ = \tanh^{-1} \frac{1}{3} - \frac{1}{2} x + \frac{x^2}{4} \]

B1  Sight of standard diffn for \( \tanh^{-1} x \)

M1  Fn of fn and diffn of quotient

A1  Sol correct quotient (i.e. correct expression for 2nd part)

A1  Correct for \( y' \)

A1  2nd diffn (NB AG)

B1  For 1st value (needs to be exact)

B1  For both

M1  Use of correct Maclaurin’s series

A1  Accept 0.347

[4]
(i) 
\[ \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \]
\[ \Rightarrow \ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \ldots \]
Validity: \(-1 \leq x \leq 1\) or \(|x| \leq 1\)

(ii) 
\[ \ln(1+x^3) = x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \ldots \]
Substitute \(x = \frac{1}{2}\)
\[ \Rightarrow \ln\left(\frac{5}{4}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\right)^4 + \frac{1}{3}\left(\frac{1}{2}\right)^6 - \frac{1}{4}\left(\frac{1}{2}\right)^8 + \ldots \]
\[ = \frac{1}{4}\left(1 - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^4 - \frac{1}{4}\left(\frac{1}{2}\right)^6 + \ldots \right) \]
\[ \Rightarrow \left(1 - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^4 - \frac{1}{4}\left(\frac{1}{2}\right)^6 + \ldots \right) \]
\[ = 4 \ln\left(\frac{5}{4}\right) \text{ i.e.w.} \]

B1 2 or 3 terms correct unsimplified
B1 All terms correct
M1 \(\frac{1}{2}\) into their ans to (i)
A1 Single ln expression
\[
\ln(1 + y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \ldots \\
\sin x = x - \frac{x^3}{6} + \ldots \\
\ln(1 + \sin x) = \left(x - \frac{x^3}{6}\right) - \frac{1}{2} \left(x - \frac{x^3}{6}\right)^2 + \frac{1}{3} \left(x - \frac{x^3}{6}\right)^3 - \ldots \\
= x - \frac{1}{2} x^2 + x^3 \left(\frac{1}{3} - \frac{1}{6}\right) \\
= x - \frac{1}{2} x^2 + \frac{1}{6} x^3
\]

Alternative using Maclaurin general formula
\[
f(x) = \ln(1 + \sin x) \\
f(0) = 0
\]
\[
f'(x) = \frac{\cos x}{1 + \sin x} \\
f'(0) = 1
\]
\[
f''(x) = \frac{-1}{(1 + \sin x)} \\
f''(0) = -1
\]
\[
f'''(x) = \frac{\cos x}{(1 + \sin x)^2} \\
f'''(0) = 1
\]

Maclaurin:
\[
f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2} + \frac{f'''(0)x^3}{6}
\]
\[
\Rightarrow f(x) = x - \frac{1}{2} x^2 + \frac{1}{6} x^3
\]

- **B1**: Soi. Allow an expansion in \( x \)
- **B1**: Soi
- **M1**: For combining series, even if wrong. Must include at least the cubic bracket.
- **A1**: Ignore further terms
  - www accept 3! for 6
- **4**
- **B1**: For \( f'(x) \)
- **B1**: For (not necessarily simplified) \( f''(x) \) and \( f'''(0) \) www
- **M1**: For correct formula up to 4th term and substituting their values
- **A1**: Accept 3! for 6
### Q9, (Jun 2016, Q5)

| (i) | \[
\frac{dy}{dx} = \frac{2}{1 + 4x^2}
\]  
\[
\frac{d^2y}{dx^2} = -2 \left(1 + 4x^2\right)^{-2} \times 8x = \frac{-16x}{\left(1 + 4x^2\right)^2} = -4x \left(\frac{dy}{dx}\right)^2
\]  
B1: For first diffn  
M1: Diffn again and making comparison  
A1: [3]  
| (ii) | When \(x = 0\), \(y = 0\), \(\frac{dy}{dx} = 2\), \(\frac{d^2y}{dx^2} = 0\)  
\[
\Rightarrow \frac{d^3y}{dx^3} + 4 \left(\frac{dy}{dx}\right)^2 + 8x \left(\frac{dy}{dx}\right) \frac{d^2y}{dx^2} = 0
\]  
B1: Soi by final answer  
M1: Differentiate the equation given  
A1: [4]  
| (iii) | \(x = \frac{1}{2} \Rightarrow \tan^{-1} 1 = \frac{\pi}{4}\)  
In series \(x = 1 - \frac{1}{3} = \frac{2}{3}\)  
\[
\Rightarrow \text{Estimate for } \pi = \frac{8}{3} = 2.666\ldots
\]  
which, correct to 1sf, = 3  
B1: soi  
B1: For showing to 1sf \(\pi = 3\) which is correct but to 2sf \(\pi = 2.7\) which is not. www