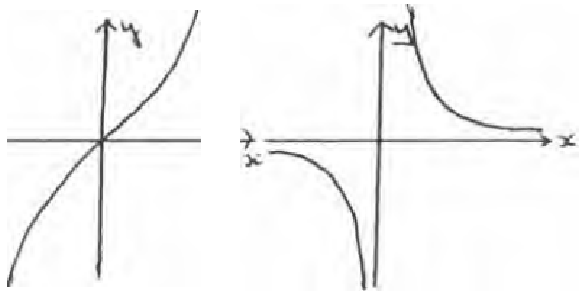
Integration Involving Trigonometric and Hyperbolic Functions (From OCR 4726)

Q1, (Jan 2007, Q4)

(i)



(i)	BI Correct shape for $\sinh x$
	B1 Correct shape for $\operatorname{cosech} x$
	B1 Obvious point $(dy/dx \neq 0)$/asymptotes clear
(ii) Correct definition of $\sinh x$	B1 May be implied
Invert and mult. by e^x to AG.	B1 Must be clear; allow $2/(e^x - e^{-x})$ as mimimum simplification
Sub. $u = e^x$ and $du = e^x\,dx$	M1 Or equivalent, all x eliminated and not $dx = du$
Replace to $2/(u^2 - 1)\,du$	Al
Integrate to $a\ln((u - 1)/(u + 1))$	A1√ Use formulae book, PT, or $\operatorname{atanh}^{-1}u$
Replace u	Al No need for c

Q2, (Jan 2008, Q9)

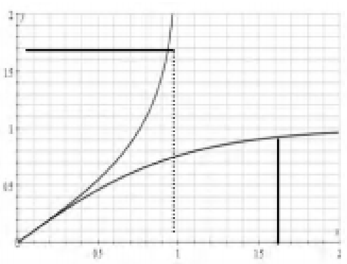
(i)	Get $\sinh y \, {}^{dy}/_{dx} = 1$	M1	Or equivalent; allow ± Allow use of ln equivalent with Chain Rule
	Replace $\sinh y = \sqrt{(\cosh^2 y - 1)}$	A1	
	Justify positive grad. to A.G.	B1	e.g. sketch
(ii)	Get $k \cosh^{-1}2x$	M1	No need for c
	Get $k=½$	A1	
(iii)	Sub. $x = k\cosh u$	M1	
	Replace all x to $\int k_1 \sinh^2 u \, du$	A1	
	Replace as $\int k_2(\cosh 2u - 1)\,du$	M1	Or exponential equivalent
	Integrate correctly	A1√	No need for c
	Attempt to replace u with x equivalent	M1	In their answer
	Tidy to reasonable form	A1	cao $(½x\sqrt{(4x^2 - 1)} - ¼\cosh^{-1}2x\ (+c))$

Q3, (Jan 2009, Q4)

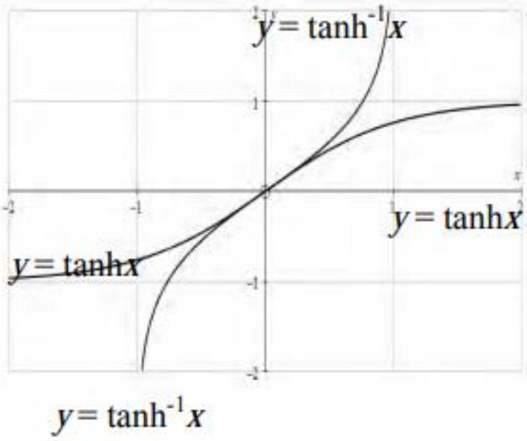
(i)	Let $x = \cosh\theta$ such that	M1		
	$dx = \sinh\theta\, d\theta$			
	Clearly use $\cosh^2 - \sinh^2 = 1$	A1	Clearly derive A.G.	
(ii)	Replace $\cosh^2\theta$	M1	Allow $a(\cosh 2\theta \pm 1)$	
	Attempt to integrate their expression	M1	Allow $b\sinh 2\theta \pm a\theta$	
	Get $\frac{1}{4}\sinh 2\theta + \frac{1}{2}\theta\ (+c)$	A1		
	Clearly replace for x to A.G.	B1	Condone no $+c$	
			SC Use expo. def^n; three terms	M1
			Attempt to integrate	M1
			Get $\frac{1}{8}(e^{2\theta} - e^{-2\theta}) + \frac{1}{2}\theta\ (+c)$	A1
			Clearly replace for x to A.G.	B1

Q4, (Jun 2009, Q6)

Get $k\sinh^{-1}k_1x$	M1	For either integral; allow attempt at ln version here
Get $\frac{1}{3}\sinh^{-1}\frac{3}{4}x$	A1	Or ln version
Get $\frac{1}{2}\sinh^{-1}\frac{2}{3}x$	A1	Or ln version
Use limits in their answers	M1	
Attempt to use correct ln laws to set up a solvable equation in a	M1	
Get $a = 2^{\frac{1}{3}}.\ 3^{\frac{1}{2}}$	A1	Or equivalent

(iii)	 Areas shown are equal: $x = \tanh k \Rightarrow y = k$	M1		For consideration of areas
		A1		For sufficient justification
	$\Rightarrow \int_0^{\tanh k} \tanh^{-1} x\, dx$	M1		For subtraction from rectangle
	$=$ rectangle $(k \times \tanh k) -$ **(ii)**	A1		For correct answer **N.B. answer given**
	$= k\tanh k - \ln(\cosh k)$		4	**Alternative:** Otherwise by parts, as $1 \times \tanh^{-1} x$ *OR* $1 \times \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

Q5, (Jun 2011, Q7)

7(i)		B1 B1 B1 B1 4	Both curves of the correct shape (ignore overlaps) and labelled gradient = 1 at $x = 0$ stated For asymptotes $y = \pm 1$ and $x = \pm 1$ (or on sketch) Sketch all correct
(ii)	$\int_0^k \tanh x \, dx = [\ln(\cosh x)]_0^k = \ln(\cosh k)$	M1 A1 2	For substituting limits into $\ln \cosh x$ For correct answer

7(iii)	Alternative method 1 By parts: $I = \int_0^{\tanh k} \tanh^{-1} x \, dx$ $u = \tanh^{-1} x \quad dv = dx$ $du = \frac{1}{1-x^2} dx \quad v = x$	**M1**	For integrating by parts (correct way round)
	$\Rightarrow I = \left[x \tanh^{-1} x \right]_0^{\tanh k} - \int_0^{\tanh k} \frac{x}{1-x^2} \, dx$	**A1** **M1**	For getting this far Dealing with the resulting integral
	$= k \tanh k + \frac{1}{2}\left[\ln(1-x^2) \right]_0^{\tanh k}$ $= k \tanh k + \frac{1}{2} \ln(1 - \tanh^2 k)$ $= k \tanh k + \frac{1}{2} \ln(\text{sech}^2 k)$ $= k \tanh k + \ln(\text{sech} k)$	**A1**	
	Alternative method 2 By substitution Let $y = \tanh^{-1} x \Rightarrow x = \tanh y$ $\Rightarrow dx = \text{sech}^2 y \, dy$ When $x = 0, \ y = 0$ When $x = \tanh k, \ y = k$	**M1**	For substitution to obtain equivalent integral
	$\Rightarrow I = \int_0^{\tanh k} \tanh^{-1} x \, dx = \int_0^k y \text{sech}^2 y \, dy$	**A1**	Correct so far
	$u = y \quad dv = \text{sech}^2 y \, dy$ $du = dy \quad v = \tanh y$	**M1**	For integration by parts (correct way round)
	$\Rightarrow I = [y \tanh y]_0^k - \int_0^k \tanh y \, dy$ $= k \tanh k - \ln \cosh k$	**A1**	Final answer

Q6, (Jun 2011, Q8)

(i)	$x=\cosh^2 u \Rightarrow \mathrm{d}u = 2\cosh u \sinh u\,\mathrm{d}u$	B1	For correct result
	$\int \sqrt{\frac{x}{x-1}}\,\mathrm{d}x = \int \frac{\cosh u}{\sinh u} 2\cosh u \sinh u\,\mathrm{d}u$	M1	For substituting throughout for x
	$= \int 2\cosh^2 u\,\mathrm{d}u$	A1	For correct simplified u integral
	$= \int (\cosh 2u + 1)\,\mathrm{d}u = \sinh u \cosh u + u$	M1	For attempt to integrate $\cosh^2 u$
		A1	For correct integration
	$= x^{\frac{1}{2}}(x-1)^{\frac{1}{2}} + \ln\left(x^{\frac{1}{2}} + (x-1)^{\frac{1}{2}}\right) (+c)$	M1	For substituting for u
		A1	For correct result
		7	**oe** as $\mathrm{f}(x)+\ln(\mathrm{g}(x))$
(ii)	$2\sqrt{3} + \ln(2+\sqrt{3})$	B1	
		1	
(iii)	$V = (\pi)\int_1^4 \frac{x}{x-1}\,\mathrm{d}x = (\pi)\left[x + \ln(x-1)\right]_1^4$	M1	For attempt to find $\int \frac{x}{x-1}\,\mathrm{d}x$
		A1	For correct integration (ignore π)
	$V \to \infty$	B1	For statement that volume is infinite (independent of M mark)
		3	

Q7, (Jan 2012, Q2)

$= \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{(2x-1)^2+4}\,dx$ *OR* $\frac{1}{4}\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{\left(x-\frac{1}{2}\right)^2+1}\,dx$	B1	For correct denominator (in 2nd case must include $\frac{1}{4}$)
	M1	For integration to $k\tan^{-1}(ax+b)$ or $k\ln\left(\frac{ax+b-c}{ax+b+c}\right)$
$= \frac{1}{2}\left[\frac{1}{2}\tan^{-1}\frac{2x-1}{2}\right]_{\frac{1}{2}}^{\frac{3}{2}}$ *OR* $\frac{1}{4}\left[\tan^{-1}\left(x-\frac{1}{2}\right)\right]_{\frac{1}{2}}^{\frac{3}{2}}$	A1	FT for $ax+b$ from their denominator For correct integration
$= \frac{1}{4}\left(\tan^{-1}1 - \tan^{-1}0\right) = \frac{1}{16}\pi$	M1	For substituting limits in any $\tan^{-1}$ expression
	A1 **[5]**	For correct value

Q8, (Jun 2012, Q1)

		$= \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{(2x-1)^2+4}\,dx$ *OR* $\frac{1}{4}\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{\left(x-\frac{1}{2}\right)^2+1}\,dx$	B1	For correct denominator (in 2nd case must include $\frac{1}{4}$)
			M1	For integration to $k\tan^{-1}(ax+b)$ or $k\ln\left(\frac{ax+b-c}{ax+b+c}\right)$
		$= \frac{1}{2}\left[\frac{1}{2}\tan^{-1}\frac{2x-1}{2}\right]_{\frac{1}{2}}^{\frac{3}{2}}$ *OR* $\frac{1}{4}\left[\tan^{-1}\left(x-\frac{1}{2}\right)\right]_{\frac{1}{2}}^{\frac{3}{2}}$	A1	FT for $ax+b$ from their denominator For correct integration
		$= \frac{1}{4}\left(\tan^{-1}1 - \tan^{-1}0\right) = \frac{1}{16}\pi$	M1	For substituting limits in any $\tan^{-1}$ expression
			A1 **[5]**	For correct value

Q9, (Jan 2013, Q6)

Working	Marks	Guidance
$x^2+4x+8=(x+2)^2+4$	M1 A1	Complete the square in order to use standard form
$\int_0^1 \frac{1}{\sqrt{x^2+4x+8}}\,dx=\int_0^1 \frac{1}{\sqrt{(x+2)^2+4}}\,dx$	M1	Use correct standard form in integration
$=\left[\sinh^{-1}\frac{x+2}{2}\right]_0^1=\sinh^{-1}\left(\frac{3}{2}\right)-\sinh^{-1}1$	A1	Answer in $\sinh^{-1}$ form
$=\ln\left(\frac{3}{2}+\sqrt{1+\frac{9}{4}}\right)-\ln\left(1+\sqrt{2}\right)=\ln\left(\frac{3}{2}+\sqrt{\frac{13}{4}}\right)-\ln\left(1+\sqrt{2}\right)$	M1	Attempt to turn into log form
$=\ln\left(\frac{3+\sqrt{13}}{2+2\sqrt{2}}\right)$	A1	www isw
	[6]	
Alternative for last 4 marks		
$\int_0^1 \frac{1}{\sqrt{(x+2)^2+4}}\,dx=\left[\ln\left((x+2)+\sqrt{(x+2)^2+4}\right)\right]_0^1$	M1 A1 M1	Attempt to use Standard form Limits
$=\ln\left(3+\sqrt{13}\right)-\ln\left(2+\sqrt{8}\right)=\ln\left(\frac{3+\sqrt{13}}{2+2\sqrt{2}}\right)$	A1	www isw
Alternative for last 4 marks		
$x+2=2\tan\theta \Rightarrow I=\left[\ln\left(\sec\theta+\tan\theta\right)\right]_{\pi/4}^{\tan^{-1}3/2}$	M1 A1	Substitution Indefinite integral
$=\ln\left(\frac{3}{2}+\frac{\sqrt{13}}{2}\right)-\ln\left(1+\sqrt{2}\right)=\ln\left(\frac{3+\sqrt{13}}{2+2\sqrt{2}}\right)$	M1 A1	Deal with limits www isw

Q10, (Jun 2014, Q1)

	$\int_0^2 \frac{1}{\sqrt{4+x^2}}\,dx = \left[\sinh^{-1}\left(\frac{x}{2}\right)\right]_0^2$	M1	Standard form	
	$= \sinh^{-1}1 - \sinh^{-1}0$	M1	Use of log form and substitute limits dep on 1st M	
	$= \ln\left(1+\sqrt{1+1}\right) - 0$	A1		
	$= \ln\left(1+\sqrt{2}\right)$ **cao isw**			
		[3]		
	Alternative: $\int_0^2 \frac{1}{\sqrt{4+x^2}}\,dx = \left[\ln\left(x+\sqrt{x^2+4}\right)\right]_0^2$	M1 M1	Standard form Substitute limits	
	$= \ln\left(2+\sqrt{8}\right) - \ln 2$			
	$= \ln\left(1+\sqrt{2}\right)$	A1		

Q11, (Jun 2015, Q3)

$\int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{2x-x^2}}\,dx = \int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{1+2x-x^2-1}}\,dx$	M1	Completing the square on given function	Or
$= \int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{1-(1-x)^2}}\,dx$	A1		$= \int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{1-(x-1)^2}}\,dx$
$= \left[-\sin^{-1}(1-x)\right]_{\frac{1}{2}}^{1}$	M1	By substitution or using standard form where completed square is of form $1-(1\pm x)^2$	$= \left[\sin^{-1}(x-1)\right]_{\frac{1}{2}}^{1} = \left(0 - -\frac{\pi}{6}\right) = \frac{\pi}{6}$
	A1	Correct result of integration. Ignore limits	
$= -\left(0-\frac{\pi}{6}\right) = \frac{\pi}{6}$	A1		
	5		

Q12, (Jun 2017, Q2)

		$I=\int_3^4 \frac{1}{x^2-6x+10}\,dx=\int_3^4 \frac{1}{(x-3)^2+1}\,dx$	**M1** **A1**	Complete the square. Denominator	Sight of $(x\pm3)^2+1$
		From formula book: $I=\left[\tan^{-1}(x-3)\right]_3^4=\tan^{-1}1\pm\tan^{-1}0$	**M1**	Use standard form and apply limits	Accept $\tan^{-1}(x\pm3)$
		$=\frac{\pi}{4}$	**A1**	www	SC correct answer only B1
			[4]		