First Order Differential Equations (From OCR 4727)

Q1, (Jun 2007, Q8)

(i) Integrating factor $e^{\int \tan x \, (dx)}$	B1		For correct IF
$= e^{-\ln \cos x}$	M1		For integrating to ln form
$= (\cos x)^{-1}$ *OR* $\sec x$	A1		For correct simplified IF **AEF**
$\Rightarrow \frac{d}{dx}\left(y(\cos x)^{-1}\right) = \cos^2 x$	B1√		For $\frac{d}{dx}(y.\text{their IF}) = \cos^3 x$. their IF
$y(\cos x)^{-1} = \int \frac{1}{2}(1+\cos 2x)\ (dx)$	M1 M1		For integrating LHS For attempting to use $\cos 2x$ formula *OR* parts for $\int \cos^2 x \, dx$
$y(\cos x)^{-1} = \frac{1}{2}x + \frac{1}{4}\sin 2x\ (+c)$	A1		For correct integration both sides **AEF**
$y = \left(\frac{1}{2}x + \frac{1}{4}\sin 2x + c\right)\cos x$	A1	**8**	For correct general solution **AEF**
(ii) $2 = \left(\frac{1}{2}\pi + c\right).-1 \Rightarrow c = -2 - \frac{1}{2}\pi$	M1		For substituting $(\pi, 2)$ into their GS and solve for c
$y = \left(\frac{1}{2}x + \frac{1}{4}\sin 2x - 2 - \frac{1}{2}\pi\right)\cos x$	A1	**2**	For correct solution **AEF**
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Q2, (Jan 2008, Q5)

(i) IF $e^{\int \frac{1}{x} dx} = e^{\ln x} = x$ *OR* $x\frac{dy}{dx} + y = x\sin 2x$	M1		For correct process for finding integrating factor *OR* for multiplying equation through by x
$\Rightarrow \frac{d}{dx}(xy) = x\sin 2x$	A1		For writing DE in this form (may be implied)
$\Rightarrow xy = \int x\sin 2x (dx)$	M1		For integration by parts the correct way round
$xy = -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x (dx)$	A1		For 1st term correct
$xy = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\ (+c)$	M1		For their 1st term and attempt at integration of $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} kx$
$\Rightarrow y = -\frac{1}{2}\cos 2x + \frac{1}{4x}\sin 2x + \frac{c}{x}$	A1	**6**	For correct expression for y
(ii) $\left(\frac{1}{4}\pi, \frac{2}{\pi}\right) \Rightarrow \frac{2}{\pi} = \frac{1}{\pi} + \frac{4c}{\pi} \Rightarrow c = \frac{1}{4}$	M1		For substituting $\left(\frac{1}{4}\pi, \frac{2}{\pi}\right)$ in solution
$\Rightarrow y = -\frac{1}{2}\cos 2x + \frac{1}{4x}\sin 2x + \frac{1}{4x}$	A1	**2**	For correct solution. Requires $\boxed{y=}$.
(iii) $(y \approx) -\frac{1}{2}\cos 2x$	B1√	**1**	For correct function **AEF** f.t. from **(ii)**
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Q3, (Jun 2009, Q4)

(i)	IF $e^{\int \frac{1}{1-x^2}dx} = e^{\frac{1}{2}\ln\frac{1+x}{1-x}} = \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$	M1 A1	2	For IF stated or implied. Allow $\pm\int$ and omission of dx For integration and simplification to **AG** (intermediate step must be seen)
(ii)	$\frac{d}{dx}\left(y\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\right) = (1+x)^{\frac{1}{2}}$	M1*		For multiplying both sides by IF
	$y\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} = \frac{2}{3}(1+x)^{\frac{3}{2}} + c$	M1 A1		For integrating RHS to $k(1+x)^n$ For correct equation (including + c)
	$(0, 2) \Rightarrow 2 = \frac{2}{3} + c \Rightarrow c = \frac{4}{3}$	M1 (*dep) M1 (*dep)		In either order: For substituting (0, 2) into their GS (including +c) For dividing solution through by IF, including dividing c or their numerical value for c
	$y = \frac{2}{3}(1+x)(1-x)^{\frac{1}{2}} + \frac{4}{3}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}$	A1	6	For correct solution aef (even unsimplified) in form $y = f(x)$

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Q4, (Jan 2010, Q3)

Integrating factor = $e^{\int 2dx} = e^{2x}$	B1		For correct IF
$\Rightarrow \frac{d}{dx}\left(ye^{2x}\right) = e^{-x}$	M1		For $\frac{d}{dx}(y.\text{their IF}) = e^{-3x}$. their IF
$\Rightarrow ye^{2x} = -e^{-x}(+c)$	A1		For correct integration both sides
$(0, 1) \Rightarrow c = 2$	M1 A1√		For substituting (0, 1) into their GS and solving for c For correct c f.t. from their GS
$\Rightarrow y = -e^{-3x} + 2e^{-2x}$	A1	6	For correct solution

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Q5, (Jan 2011, Q1)

(i)	Integrating factor. $e^{\int x\,dx} = e^{\frac{1}{2}x^2}$	B1		For correct IF
	$\Rightarrow \frac{d}{dx}\left(ye^{\frac{1}{2}x^2}\right) = xe^{x^2}$	M1		For $\frac{d}{dx}(y.\text{their IF}) = xe^{\frac{1}{2}x^2}$.their IF
	$\Rightarrow ye^{\frac{1}{2}x^2} = \frac{1}{2}e^{x^2}(+c)$	A1		For correct integration both sides
	$\Rightarrow y = e^{-\frac{1}{2}x^2}\left(\frac{1}{2}e^{x^2} + c\right) = \frac{1}{2}e^{\frac{1}{2}x^2} + ce^{-\frac{1}{2}x^2}$	A1	4	For correct solution **AEF** as $y = f(x)$
(ii)	$(0, 1) \Rightarrow c = \frac{1}{2}$	M1		For substituting (0, 1) into their GS, solving for c and obtaining a solution of the DE
	$\Rightarrow y = \frac{1}{2}\left(e^{\frac{1}{2}x^2} + e^{-\frac{1}{2}x^2}\right)$	A1	2	For correct solution **AEF** Allow $y = \cosh\left(\frac{1}{2}x^2\right)$

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Q6, (Jun 2011, Q3)

(i)	METHOD 1 $m+4\ (=0) \Rightarrow$ CF $(y=)\,Ae^{-4x}$	M1 A1 **2**	For correct auxiliary equation (soi) For correct CF
	METHOD 2 Separating variables on $\frac{dy}{dx}+4y=0$		
	$\Rightarrow \ln y = -4x$	M1	For integration to this stage
	$\Rightarrow$ CF $(y=)\,Ae^{-4x}$	A1	For correct CF
(ii)	PI $(y=)\ p\cos 3x + q\sin 3x$	B1	For stating PI of correct form
	$y' = -3p\sin 3x + 3q\cos 3x$	M1	For substituting y and y' into DE
	$\Rightarrow (-3p+4q)\sin 3x + (4p+3q)\cos 3x = 5\cos 3x$	A1	For correct equation
	$\Rightarrow \left.\begin{matrix}-3p+4q=0\\ 4p+3q=5\end{matrix}\right\} \Rightarrow p=\frac{4}{5},\ q=\frac{3}{5}$	M1 A1 A1	For equating coeffs and solving For correct value of p, and of q
	GS $(y=)\ Ae^{-4x}+\frac{4}{5}\cos 3x+\frac{3}{5}\sin 3x$	B1√ **7**	For GS f.t. from their CF+PI with 1 arbitrary constant in CF and none in PI
	SR Integrating factor method may be used, followed by 2-stage integration by parts or C+iS method Marks for **(i)** are awarded only if CF is clearly identified		
(iii)	$e^{-4x}\to 0$, $\frac{4}{5}\cos 3x+\frac{3}{5}\sin 3x = \frac{\sin}{\cos}(3x+\alpha)$	M1	For considering either term
	$\Rightarrow -1 \leqslant y \leqslant 1$ *OR* $-1 \lessapprox y \lessapprox 1$	A1√ **2**	For correct range (allow <) **CWO** f.t. as $-\sqrt{p^2+q^2} \leqslant y \leqslant \sqrt{p^2+q^2}$ from **(ii)**
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Q7, (Jun 2012, Q3)

Integrating factor = $e^{\int \cot x\,dx} = e^{\ln \sin x} = \sin x$	M1	For IF = $e^{\pm \ln \sin x}$ *OR* $e^{\pm \ln \cos x}$	
	A1	For simplified IF	
$\Rightarrow \frac{d}{dx}(y \sin x) = 2x \sin x$	M1	For $\frac{d}{dx}(y.\text{their IF}) = 2x.\text{their IF}$	
$\Rightarrow y \sin x = -2x \cos x + \int 2 \cos x\,dx$	M1*	For attempt to integrate RHS using parts for $\int x \begin{cases} \sin x \\ \cos x \end{cases} dx$	(Must use u = (2)x)
	A1	For correct RHS 1st stage	
$\Rightarrow y \sin x = -2x \cos x + 2 \sin x\ (+c)$	A1	oe	
$\left(\frac{1}{6}\pi, 2\right) \Rightarrow c = \frac{1}{6}\pi\sqrt{3}$	M1dep *	For substituting $\left(\frac{1}{6}\pi, 2\right)$ into their GS (with c)	c = 0.907
	A1 FT	For correctly finding c (FT from GS)	
$\Rightarrow y = -2x \cot x + 2 + \frac{1}{6}\pi\sqrt{3} \operatorname{cosec} x$	A1	For correct solution **AEF** of standard notation $y = f(x)$	
	[9]		

Q8, (Jan 2013, Q3)

$\frac{dy}{dx} - 3\frac{y}{x} = x^3 e^{2x}$	M1	Divide by x
$I = \exp\left(\int -\frac{3}{x}dx\right) = e^{-3\ln x}$	M1	
$= x^{-3}$	A1	
$x^{-3}\frac{dy}{dx} - 3x^{-4}y = e^{2x}$	M1	Multiply and recognise derivative
$\frac{d}{dx}\left(x^{-3}y\right) = e^{2x}$	M1	Integrate
$x^{-3}y = \frac{1}{2}e^{2x} + A$	A1	
$x = 1, y = 0 \Rightarrow A = -\frac{1}{2}e^2$	M1	Use condition
$y = \frac{1}{2}x^3\left(e^{2x} - e^2\right)$	A1	
	[8]	

Q9, (Jun 2013, Q3)

$u = y^3 \Rightarrow \dfrac{du}{dx} = 3y^2\dfrac{dy}{dx}$	M1		Or $\dfrac{dy}{dx} = \dfrac{1}{3}u^{-\frac{2}{3}}\dfrac{du}{dx}$
in DE gives $x\dfrac{du}{dx} + 2u = \dfrac{\cos x}{x}$	A1		
$\dfrac{du}{dx} + \dfrac{2}{x}u = \dfrac{\cos x}{x^2}$	B1	Divide	Both sides
$I = \exp\left(\int \frac{2}{x}dx\right) = e^{2\ln x}$	M1	Correctly integrates	Must have form $\dfrac{du}{dx} + f(x)u = g(x)$
$= x^2$	A1		Can be implied by subsequent work
$x^2\dfrac{du}{dx} + 2xu = \cos x$			
$\dfrac{d}{dx}\left(x^2u\right) = \cos x$			
$x^2u = \sin x \quad (+A)$	M1	Integrate	
$u = \dfrac{\sin x + A}{x^2}$	A1	Or gives GS in implicit form	Must include constant at this stage
$y = \left(\dfrac{\sin x + A}{x^2}\right)^{\frac{1}{3}}$	A1		
	[8]		

Q10, (Jun 2014, Q2)

$u = y^2 \Rightarrow \frac{du}{dx} = 2y\frac{dy}{dx}$	M1	Correctly finds	Or $\frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}}\frac{du}{dx}$
so DE $\Rightarrow 2y\frac{dy}{dx} - 4y^2 = 2e^x$	M1	or for complete unsimplified substitution	
$\Rightarrow \frac{du}{dx} - 4u = 2e^x$	A1		Can be implied by next A1
$I = \exp\int -4\,dx = e^{-4x}$	A1ft		Must have form $\frac{du}{dx} + f(x)u = g(x)$ for this mark and any further marks Can be implied by subsequent work
$e^{-4x}\frac{du}{dx} - 4e^{-4x}u = 2e^{-3x}$	M1*	Multiples through by IF of form e^{kx}, simplifying RHS	
$ue^{-4x} = -\frac{2}{3}e^{-3x}(+A)$	*M1dep*	Integrates	
$u = -\frac{2}{3}e^x + Ae^{4x}$	M1dep *	Rearranges to make u or y^2 the subject	No more than 1 numerical error at this step
$y = \sqrt{-\frac{2}{3}e^x + Ae^{4x}}$	A1	Cao	ignore use of '±'
Alternative from 4th mark to 6th mark			
CF: (u=…) Ae^{4x}	A1		
PI: $u = ke^x$, $\frac{du}{dx} = ke^x$	M1*	PI chosen & differentiated correctly	
$ke^x - 4ke^x = 2e^x$, $k = -\frac{2}{3}$	M1 dep*	Substitutes and solves	
	[8]		

Q11, (Jun 2015, Q5)

$\frac{dy}{dx}+\frac{3}{x}y=x+1$	B1	Divide both sides by x	
$I=\exp\left(\int \frac{3}{x}dx\right)=e^{3\ln x}$	M1		
$=x^3$	A1		A0 means no further marks can be gained
$x^3\frac{dy}{dx}+3x^2y=x^4+x^3$			
$\frac{d}{dx}\left(x^3y\right)=\cdots$ $\cdots=x^4+x^3$	M1	Multiply and recognise derivative	
	M1	Integrate both sides (their two term polynomial)	condone absent A at this stage
$x^3y=\frac{1}{5}x^5+\frac{1}{4}x^4+A$	A1		
$x=1, y=1\Rightarrow A=\frac{11}{20}$	M1	Use condition	
$y=\frac{1}{5}x^2+\frac{1}{4}x+\frac{11}{20}x^{-3}$	A1		
	[8]		

Q12, (Jun 2017, Q1)

$(I=)\exp(\int \cot x\,dx)$	M1		
$=e^{\ln\sin x}$	M1		
$=\sin x$	A1		
$\frac{d}{dx}(y\sin x)=9$	M1*	Multiply and integrate	
$y\sin x=9x+A$	A1		
$x=\frac{1}{6}\pi, y=\pi\Rightarrow\frac{1}{2}\pi=\frac{3}{2}\pi+A\Rightarrow A=-\pi$	M1	Correct substitution of given point and constant evaluated	
$y=(9x-\pi)\operatorname{cosec}x$	*M1dep	Rearrange to isolate "y"	
	A1	oe	Must have "y ="
	[8]		