Q1, (Jun 2007, Q8)

(i) Integrating factor \( e^{\tan x} (dx) \)

\[ e^{-\ln \cos x} = (\cos x)^{-1} OR \sec x \]

\[ \frac{d}{dx}(y(\cos x)^{-1}) = \cos^2 x \]

\[ y(\cos x)^{-1} = \int \frac{1}{2} (1 + \cos 2x) \,(dx) \]

\[ y(\cos x)^{-1} = \frac{1}{2} x + \frac{1}{4} \sin 2x + c \]

\[ y = \left( \frac{1}{2} x + \frac{1}{4} \sin 2x + c \right) \cos x \]

B1  For correct IF
M1  For integrating to ln form
A1  For correct simplified IF AEF
B1√ For \( \frac{d}{dx}(y, \text{their IF}) = \cos^3 x \), their IF
M1  For integrating LHS
M1  For attempting to use \( \cos 2x \) formula OR parts for \( \int \cos^2 x \, dx \)
A1  For correct integration both sides AEF
A1 8 For correct general solution AEF

(ii) \( 2 = \left( \frac{1}{2} \pi + c \right) \)

\[ -1 \Rightarrow c = -2 - \frac{1}{2} \pi \]

\[ y = \left( \frac{1}{2} x + \frac{1}{4} \sin 2x - 2 - \frac{1}{2} \pi \right) \cos x \]

M1  For substituting \( (\pi, 2) \) into their GS and solve for \( c \)
A1 2 For correct solution AEF

Q2, (Jan 2008, Q5)

(i) IF \( e^{\int \frac{1}{x} \, dx} = e^{\ln x} = x \)

\[ OR \quad x \frac{dy}{dx} + y = x \sin 2x \]

\[ \Rightarrow \frac{d}{dx}(xy) = x \sin 2x \]

\[ \Rightarrow xy = \int x \sin 2x \,(dx) \]

\[ xy = -\frac{1}{2} x \cos 2x + \frac{1}{4} \int \cos 2x \,(dx) \]

\[ xy = -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + c \]

\[ \Rightarrow y = -\frac{1}{2} \cos 2x + \frac{1}{4} \sin 2x + \frac{c}{x} \]

M1  For correct process for finding integrating factor OR for multiplying equation through by \( x \)
A1  For writing DE in this form (may be implied)
M1  For integration by parts the correct way round
A1  For 1st term correct
M1  For their 1st term and attempt at integration of \( \frac{\cos 2x}{\sin^2 x} \)
A1  6 For correct expression for \( y \)

(ii) \( \left( \frac{1}{4}, \frac{\pi}{2} \right) \)

\[ \Rightarrow \frac{2}{\pi} = -\frac{1}{4} + \frac{4c}{\pi} \Rightarrow c = \frac{1}{4} \]

\[ \Rightarrow y = -\frac{1}{2} \cos 2x + \frac{1}{4} \sin 2x + \frac{1}{4x} \]

M1  For substituting \( \left( \frac{1}{4}, \frac{\pi}{2} \right) \) in solution
A1  2 For correct solution. Requires \( y = \)

(iii) \( y \approx -\frac{1}{2} \cos 2x \)

B1√  1 For correct function AEF f.t. from (ii)
9
Q3, (Jun 2009, Q4)

(i) \[ \text{IF } e^{\frac{1}{2} \ln \frac{1+x}{1-x}} = e^{\frac{1}{2} \ln \frac{1+x}{1-x}} = \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} \]

\[ M1 \quad \text{For IF stated or implied. Allow } \int \text{ and omission of } dx \]

\[ A1 \quad 2 \quad \text{For integration and simplification to AG (Intermediate step must be seen)} \]

(ii) \[ \frac{d}{dx}\left(y \left(1+x\right)^{\frac{1}{2}} \left(1-x\right)^{\frac{1}{2}}\right) = (1+x)^{\frac{1}{2}} \]

\[ M1^* \quad \text{For multiplying both sides by IF} \]

\[ y\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} = \frac{2}{3} (1+x)^{\frac{3}{2}} + c \]

\[ M1 \quad \text{For integrating RHS to } k(1+x)^n \]

\[ A1 \quad \text{For correct equation (including } + c \text{)} \]

\[ \text{In either order:} \]

\[ M1 \quad \text{For substituting } (0, 2) \text{ into their GS (including } + c) \]

\[ (*\text{dep}) \]

\[ M1 \quad \text{For dividing solution through by IF,} \]

\[ (*\text{dep}) \quad \text{including dividing } c \text{ or their numerical value for } c \]

\[ A1 \quad 6 \quad \text{For correct solution} \]

\[ \text{aef (even unsimplified) in form } y = f(x) \]

Q4, (Jan 2010, Q3)

Integrating factor = \( e^{\int 2dx} = e^{2x} \)

\[ B1 \quad \text{For correct IF} \]

\[ M1 \quad \text{For } \frac{d}{dx}(y \text{ their IF}) = e^{-3x} \text{ their IF} \]

\[ A1 \quad \text{For correct integration both sides} \]

\[ M1 \quad \text{For substituting } (0, 1) \text{ into their GS and solving for } c \]

\[ A1^\vee \quad \text{For correct } c \text{ f.t. from their GS} \]

\[ A1 \quad 6 \quad \text{For correct solution} \]

Q5, (Jan 2011, Q1)

(i) \[ \text{Integrating factor. } e^{\frac{1}{2} \ln x} = e^{\frac{1}{2} x^2} \]

\[ B1 \quad \text{For correct IF} \]

\[ M1 \quad \text{For } \frac{d}{dx}(y \text{ their IF}) = xe^{\frac{1}{2} x^2} \text{ their IF} \]

\[ A1 \quad \text{For correct integration both sides} \]

\[ A1 \quad 4 \quad \text{For correct solution AEF as } y = f(x) \]

(ii) \[ (0, 1) \Rightarrow c = \frac{1}{2} \]

\[ \Rightarrow y = \frac{1}{2} \left( e^{\frac{1}{2} x^2} + e^{-\frac{1}{2} x^2} \right) \]

\[ M1 \quad \text{For substituting } (0, 1) \text{ into their GS, solving for } c \text{ and obtaining a solution of the DE} \]

\[ A1 \quad 2 \quad \text{For correct solution AEF Allow } y = \cosh \left( \frac{1}{2} x^2 \right) \]

\[ 6 \]

\[ 6 \]
Q6 (Jun 2011, Q3)

(i) METHOD 1

\[ m + 4 ( = 0) \Rightarrow CF \ (y =) \ Ae^{-4x} \]

M1 For correct auxiliary equation (soi)

A1 2 For correct CF

METHOD 2

Separating variables on \( \frac{dy}{dx} + 4y = 0 \)

\[ \Rightarrow \ln y = -4x \]

\[ \Rightarrow CF \ (y =) \ Ae^{-4x} \]

M1 For integration to this stage

A1 For correct CF

(ii) PI \ (y =) \ p \cos 3x + q \sin 3x

\[ y' = -3p \sin 3x + 3q \cos 3x \]

\[ \Rightarrow (-3p + 4q) \sin 3x + (4p + 3q) \cos 3x = 5 \cos 3x \]

\[ \Rightarrow -3p + 4q = 0 \]

\[ 4p + 3q = 5 \]

\[ \Rightarrow p = \frac{4}{5}, \ q = \frac{3}{5} \]

M1 For substituting \( y \) and \( y' \) into DE

A1 For correct equation

A1 A1 For equating coeffs and solving

B1√ 7 For correct value of \( p \), and of \( q \)

GS \ (y =) \ Ae^{-4x} + \frac{4}{5} \cos 3x + \frac{3}{5} \sin 3x

B1√ 7 For GS

f.t. from their CF + PI with 1 arbitrary constant in CF and none in PI

SR Integrating factor method may be used, followed by 2-stage integration by parts or \( C+iS \) method

Marks for (i) are awarded only if CF is clearly identified

(iii) \[ e^{-4x} \to 0, \ \frac{4}{5} \cos 3x + \frac{3}{5} \sin 3x = \frac{\sin (3x + \alpha)}{\cos (3x + \alpha)} \]

\[ \Rightarrow -1 \leq y \leq 1 \quad OR \quad -1 \leq y \leq 1 \]

M1 For considering either term

A1√ 2 For correct range (allow <) CWO

f.t. as \( -\sqrt{p^2 + q^2} \leq y \leq \sqrt{p^2 + q^2} \) from (ii)

[11]
### Q7 (Jun 2012, Q3)

Integrating factor = \( e^{\cot x \, dx} = e^{\ln \sin x} = \sin x \)

\[
\Rightarrow \frac{d}{dx}(y \sin x) = 2x \sin x
\]

\[
\Rightarrow y \sin x = -2x \cos x + \int 2 \cos x \, dx
\]

\[
\Rightarrow y \sin x = -2x \cos x + 2 \sin x + c
\]

\[
\left(\frac{1}{6} \pi, 2\right) \Rightarrow c = \frac{1}{6} \pi \sqrt{3}
\]

\[
\Rightarrow y = -2x \cot x + 2 + \frac{1}{6} \pi \sqrt{3} \csc x
\]

- **M1**: For IF = \( e^{\pm \ln \sin x} \) OR \( e^{+ \ln \cos x} \)
- **A1**: For simplified IF
- **M1**: For \( \frac{d}{dx}(y \text{their IF}) = 2x \text{ their IF} \)
- **M1\text{*}**: For attempt to integrate RHS using parts for \( \int x \left(\frac{\sin x}{\cos x}\right) \, dx \)
- **A1**: For correct RHS 1st stage
- **A1**: oe
- **M1\text{dep*}**: For substituting \( \left(\frac{1}{6} \pi, 2\right) \) into their GS (with c)
- **A1 FT**: For correctly finding c (FT from GS)
- **A1**: For correct solution AEF of standard notation \( y = f(x) \)

\[\text{[9]}\]

### Q8 (Jan 2013, Q3)

\[
\frac{dy}{dx} - 3y - x = x^3 \, e^{2x}
\]

\[
l = \exp \left( \int -\frac{3}{x} \, dx \right) = e^{-3 \ln x}
\]

\[
= x^{-3}
\]

\[
x^{-3} \frac{dy}{dx} - 3x^{-4} y = e^{2x}
\]

\[
\frac{d}{dx} \left( x^{-3} y \right) = e^{2x}
\]

\[
x^{-3} y = \frac{1}{2} e^{2x} + A
\]

\[
x = 1, y = 0 \Rightarrow A = -\frac{1}{2} e^2
\]

\[
y = \frac{1}{2} x^3 \left( e^{2x} - e^2 \right)
\]

- **M1**: Divide by x
- **M1**: Multiply and recognise derivative
- **A1**: Integrate
- **M1**: Use condition

\[\text{[8]}\]
\[ q9 \quad (Jun \quad 2013, \quad Q3) \]

\[ u = y^3 \Rightarrow \frac{du}{dx} = 3y^2 \frac{dy}{dx} \]

in DE gives \( x \frac{du}{dx} + 2u = \frac{\cos x}{x} \)

\[ \frac{du}{dx} + \frac{2}{x} u = \frac{\cos x}{x^2} \]

\[ I = \exp \left( \int \frac{2}{x} \, dx \right) = e^{2 \ln x} \]

\[ = x^2 \]

\[ x^2 \frac{du}{dx} + 2xu = \cos x \]

\[ \frac{d}{dx} \left( x^2 u \right) = \cos x \]

\[ x^2 u = \sin x \quad (+A) \]

\[ u = \frac{\sin x + A}{x^2} \]

\[ y = \left( \frac{\sin x + A}{x^2} \right)^{\frac{1}{3}} \]

M1

A1

B1 Divide

Both sides

M1 Correctly integrates

Must have form \( \frac{du}{dx} + f(x)u = g(x) \)

A1 Can be implied by subsequent work

M1 Integrate

A1 Or gives GS in implicit form

Must include constant at this stage

[8]
Q10, (Jun 2014, Q2)

\[ u = y^2 \Rightarrow \frac{du}{dx} = 2y \frac{dy}{dx} \]

so DE \( \Rightarrow 2y \frac{dy}{dx} - 4y^2 = 2e^x \)

\[ \Rightarrow \frac{du}{dx} - 4u = 2e^x \]

\[ I = \exp \int -4dx = e^{-4x} \]

\[ e^{-4x} \frac{du}{dx} - 4e^{-4x} u = 2e^{-3x} \]

\[ u e^{-4x} = -\frac{2}{3} e^{-3x} (A) \]

\[ u = -\frac{2}{3} e^x + Ae^{4x} \]

\[ y = \sqrt{-\frac{2}{3} e^x + Ae^{4x}} \]

Alternative from 4th mark to 6th mark

CF: \( u = \ldots \) \( Ae^{4x} \)

PI: \( u = ke^x, \frac{du}{dx} = ke^x \)

\[ ke^x - 4ke^x = 2e^x, \quad k = -\frac{2}{3} \]

<table>
<thead>
<tr>
<th>M1</th>
<th>Correctly finds</th>
<th>Or ( \frac{dy}{dx} = \frac{1}{2u} ) ( \frac{1}{2} \frac{du}{dx} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>or for complete unsimplified substitution</td>
<td>Can be implied by next A1</td>
</tr>
<tr>
<td>A1</td>
<td></td>
<td>Must have form ( \frac{du}{dx} + f(x)u = g(x) ) for this mark and any further marks Can be implied by subsequent work</td>
</tr>
<tr>
<td>A1</td>
<td></td>
<td>No more than 1 numerical error at this step ignore use of ‘±’</td>
</tr>
</tbody>
</table>

| M1* | Multiplies through by IF of form \( e^{kx} \), simplifying RHS |                             |
| *M1dep* | Integrates |                             |
| M1dep* | Rearranges to make \( u \) or \( y^2 \) the subject |                             |
| A1 | Cao |                             |

| A1 |                             |                             |
| M1* | PI chosen & differentiated correctly |                             |
| M1 dep* | Substitutes and solves |                             |

[8]
Q11, (Jun 2015, Q5)
\[ \frac{dy}{dx} + \frac{3}{x} y = x + 1 \]
\[ l = \exp\left( \int \frac{1}{x} \, dx \right) = e^{3\ln x} = x^3 \]
\[ x^3 \frac{dy}{dx} + 3x^2 y = x^4 + x^3 \]
\[ \frac{d}{dx} \left( x^3 y \right) = \ldots \]
\[ \ldots = x^4 + x^3 \]
\[ x^3 y = \frac{1}{3} x^5 + \frac{1}{4} x^4 + A \]
\[ x=1, y=1 \Rightarrow A = \frac{11}{20} \]
\[ y = \frac{1}{3} x^2 + \frac{1}{4} x + \frac{11}{20} x^3 \]

Q12, (Jun 2017, Q1)
\[ (l = ) \exp\left( \int \cot x \, dx \right) \]
\[ = e^{\ln \sin x} = \sin x \]
\[ \frac{d}{dx} (y \sin x) = 9 \]
\[ y \sin x = 9x + A \]
\[ x = \frac{1}{6} \pi, y = \pi \Rightarrow \frac{1}{2} \pi = \frac{3}{2} \pi + A \Rightarrow A = -\pi \]
\[ y = (9x - \pi) \csc x \]

| B1 | Divide both sides by \( x \) |
| M1 | Multiply and recognise derivative |
| A1 | Integrate both sides (their two term polynomial) |
|     | Condone absent \( A \) at this stage |
| A1 | Use condition |
| [8] |

| M1 | Multiply and integrate |
| A1 | Correct substitution of given point and constant evaluated |
| *M1dep | Rearrange to isolate “y” |
| A1 | o.e. |
| [8] | Must have “y =” |