Differentiation of Inverse Trigonometric and Hyperbolic Functions (From OCR 4726)

Q1, (Jun 2006, Q2)
(i) Get \( \sec^2 y \frac{dy}{dx} = 1 \) or equivalent \( \frac{dy}{dx} \)
Clearly use \( 1 + \tan^2 y = \sec^2 y \)
Clearly arrive at A.G.

(ii) Reasonable attempt to diff. to \( \frac{-2x}{(1+x^2)^2} \)
Substitute their expressions into D.E.
Clearly arrive at A.G.

M1
M1 May be implied
A1

M1 Use of chain/quotient rule

M1 Or attempt to derive diff. equ^n.
A1
SC Attempt diff. of \( (1+x^2)\frac{dy}{dx} = 1 \)
M1,A1

Clearly arrive at A.G. B1

Q2, (Jun 2007, Q4)
(i) Reasonable attempt at product rule
Derive or quote diff. of \( \cos^{-1} x \)
Get \(-x^2 (1-x^2)^{\frac{1}{2}} + (1-x^2)^{\frac{1}{2}} + (1-x^2)^{-\frac{1}{2}} \)
Tidy to \( 2(1-x^2)^{\frac{1}{2}} \)

M1 Two terms seen
M1 Allow +
A1
A1 cao

(ii) Write down integral from (i)
Use limits correctly
Tidy to \( \frac{1}{2}\pi \)

B1 On any \( k\sqrt{1-x^2} \)
M1 In any reasonable integral
A1
SR Reasonable sub. B1
Replace for new variable and attempt to integrate (ignore limits)
M1
Clearly get \( \frac{1}{2}\pi \)
A1

Q3, (Jan 2008, Q9i)
(i) Get sinh \( y \frac{dy}{dx} = 1 \)

M1 Or equivalent; allow ±
Allow use of ln equivalent with Chain Rule

Replace sinh \( y = \sqrt{\cosh^2 y - 1} \)
A1

Justify positive grad. to A.G. B1 e.g. sketch
Q4, (Jan 2010, Q9)

(i) Rewrite \( \tanh y = (e^y - e^{-y})/(e^y + e^{-y}) \) as quadratic in \( e^{2y} \) 
Attempt to write as quadratic in \( e^{2y} \) 
Clearly get A.G. 
B1 Or equivalent

(ii) (a) Attempt to diff. and solve \( = 0 \) 
Get \( \tanh x = b/a \) 
Use \( -1 < \tanh x < 1 \) to show \( b < a \) 
M1  A1

SC Use exponentials 
Get \( e^{2x} = (a + b)/(a - b) \) 
Use \( e^{2x} > 0 \) to show \( b < a \) 
M1 A1 B1

SC Write \( x = \tanh^{-1}(b/a) \) 
M1
\[ = \frac{1}{2}\ln((1 + b/a)/(1 - b/a)) \]
A1
Use \( () > 0 \) to show \( b < a \) 
B1

(b) Get \( x = 1/a \) from part (ii)(a) 
Replace as In from their answer 
Get \( x = \frac{1}{2}\ln((a + 1)/(a - 1)) \) 
Use \( e^{\frac{1}{2}\ln((a+1)/(a-1))} = \sqrt{(a + 1)/(a - 1)} \) 
M1 A1
Clearly get A.G. 
A1
Test for minimum correctly 
B1

SC Use of \( y = \cosh x \) \((a - \tanh x)\) and 
\( \cosh x = 1/\text{sech} x = 1/\sqrt{1 - \tanh^2 x} \)

Q5, (Jun 2010, Q6)

(i) Reasonable attempt to differentiate 
\( \sinh y = x \) to get \( dy/dx \) in terms of \( y \) 
Replace \( \sinh y \) to A.G. 
M1 Allow \( \pm \cosh y \) \( dy/dx = 1 \) 
A1 Clearly use \( \cosh^2 - \sinh^2 = 1 \) 
SC Attempt to diff. \( y = \ln(x + \sqrt{x^2 + 1}) \) 
using chain rule 
M1 
Clearly tidy to A.G. 
A1

(ii) Reasonable attempt at chain rule 
Get \( dy/dx = a \sinh(\sinh^{-1} x)/\sqrt{x^2 + 1} \) 
M1 To give a product 
A1
Reasonable attempt at product/quotient 
Get \( d^2 y/dx^2 \) correctly in some form 
M1 Must involve \( \sinh \) and \( \cosh \) 
A1\sqrt From \( dy/dx = k \sinh(\sinh^{-1} x)/\sqrt{x^2 + 1} \) 
A1
SC Write \( \sqrt{(x^2 + 1)} dy/dx = k \sinh(\sinh^{-1} x) \) or similar 
Derive the A.G.
(i) $x = \sin y \Rightarrow \frac{dx}{dy} = \cos y$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$

$+ \sqrt{\text{ taken since } \sin^{-1} x \text{ has positive gradient}}$

M1 For implicit diffn $\frac{dy}{dx} = \pm \frac{1}{\cos y}$

A1 For using $\sin^2 y + \cos^2 y = 1$ to obtain N.B. Answer given

B1 For justifying + sign

(ii) $f(0) = 0, \ f'(0) = 1$

$\Rightarrow f''(x) = \frac{x}{\left(1 - x^2\right)^{3/2}}$

$\Rightarrow f''(x) = \frac{(1-x^2)^{3/2} + 3x^2(1-x^2)^{1/2}}{(1-x^2)^3}$

$\Rightarrow f''(0) = 0, \ f''(0) = 1$

$\Rightarrow \sin^{-1} x = x + \frac{1}{6} x^3$

B1 For correct values

M1 Use of chain rule to differentiate $f'(x)$

M1 Use of quotient or product rule to differentiate $f''(0)$.

A1 For correct values www, soi

A1 For correct series (allow 3!) www

Alternative Method:

$f(0) = 0, f'(0) = 1$

$\Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}} = \left(1-x^2\right)^{-1/2} = 1 + \frac{1}{2} x^2 + \frac{3}{8} x^4 + ...$

$\Rightarrow f''(x) = x + \frac{3}{2} x^3 + ...$

$\Rightarrow f''(x) = 1 + \frac{9}{2} x^2 + ...$

$\Rightarrow f''(0) = 1, f''(0) = 0, f''(0) = 1$

$\Rightarrow \sin^{-1} x = x + \frac{1}{6} x^3$

B1 For correct values

M1 Correct use of binomial

M1 Differentiate twice

A1 Correct values

A1 Correct series

(iii) $(\sin^{-1} x) \ln(1+x)$

$= \left(x + \frac{1}{6} x^3\right) \left(x - \frac{1}{2} x^2 + \frac{1}{3} x^3\right)$

$= x^2 - \frac{1}{2} x^3 + \frac{1}{2} x^4$

B1ft For terms in both series to at least $x^3$

f.t. from their (ii) multiplied together

M1 For multiplying terms to at least $x^3$

A1 For correct series up to $x^3$ www

A1 For correct term in $x^4$ www
\( \cos y = x \Rightarrow -\sin y \frac{dy}{dx} = 1 \)
\[
\Rightarrow \frac{dy}{dx} = \frac{-1}{\sin y} = \frac{1}{\sqrt{1-x^2}}
\]
- sign since \( \frac{dy}{dx} < 0 \) (e.g. by graph)

\[
\frac{dy}{dx} = \frac{-2x}{\sqrt{1-(1-x^2)^2}}
\]
\[
= \frac{2x}{\sqrt{2-x^2}} = \frac{2}{\sqrt{2-x^2}}
\]
\[
\frac{d^2y}{dx^2} = 2 \cdot \frac{1}{2} \cdot 2x (2-x^2)^{-\frac{3}{2}} = \frac{2x}{(2-x^2)^{\frac{3}{2}}}
\]
\[
\Rightarrow (2-x^2) \frac{d^2y}{dx^2} = \frac{2x}{\sqrt{2-x^2}} = x \frac{dy}{dx}
\]
\[ \frac{dy}{dx} = \frac{1}{1 - \left( \frac{1-x}{3+x} \right)^2} \times \frac{-(3+x) - (1-x)}{(3+x)^2} \]

\[ \Rightarrow \frac{dy}{dx} = \left( \frac{-4}{(3+x)^2 - (1-x)^2} \right) = \frac{k}{1+x} \]

\[ \Rightarrow \frac{dy}{dx} = \frac{-1}{2(1+x)} \]

\[ \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2(1+x)^2} \]

(i) Sight of standard diffn for \( \tanh^{-1} x \)

Fn of fn and diffn of quotient

Soi correct quotient (i.e. correct expression for 2nd part)

Correct for \( y' \)

2nd diffn (NB AG)

[6]

(ii) When \( x = 0, y = \tanh^{-1} \frac{1}{3} \) or \( \frac{1}{2} \ln 2 \) or \( \ln \sqrt{2} \)

\[ \frac{dy}{dx} = -\frac{1}{2} \]

\[ \frac{d^2y}{dx^2} = \frac{1}{2} \]

\[ \Rightarrow y = \tanh^{-1} \frac{1}{3} + \left( -\frac{1}{2} \right) x + \left( \frac{1}{2} \right) \frac{x^2}{2} \]

\[ = \tanh^{-1} \frac{1}{3} - \frac{1}{2} x + \frac{x^2}{4} \]

For 1st value (needs to be exact)

For both

Use of correct Maclaurin’s series

Accept 0.347

[4]
(i) For 1st curve $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$
For 2nd curve $\tan^{-1}\left(\sqrt{2} \times \frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

(ii) For 1st curve $y = \cos^{-1} x$, \[
\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}
\]
For 2nd curve $y = \tan^{-1} x$, \[
\frac{dy}{dx} = \frac{\sqrt{2}}{1 + 2x^2}
\]
For 1st curve, when $x = \frac{1}{\sqrt{2}}$, \[
\frac{dy}{dx} = \frac{-1}{\sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2}} = \frac{-1}{\frac{1}{\sqrt{2}}} = -\sqrt{2}
\]
For 2nd curve, when $x = \frac{1}{\sqrt{2}}$, \[
\frac{dy}{dx} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}
\]

Since $m_1 \times m_2 = -1$ then Yes

Acceptable reason:
One the negative reciprocal of the other.
Condone: One the negative inverse of the other
### Q10, (Jun 2014, Q6)

#### (i)

\[ x = \cosh y = \frac{e^y + e^{-y}}{2} \Rightarrow e^y + e^{-y} = 2x \]
\[ \Rightarrow e^{2y} - 2xe^y + 1 = 0 \]
\[ \Rightarrow e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1} \]
\[ \Rightarrow y = \ln x \pm \sqrt{x^2 - 1} \]

Reject the sign as principal value taken
\[ \Rightarrow y = \ln x + \sqrt{x^2 - 1} \]

- **M1**: Finding 3 term quadratic in $e^y$
- **A1**: Correct solution
- **B1**: Including reason oe

#### (ii)

\[ y = \ln x + \sqrt{x^2 - 1} \Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 - 1}} \times \left( 1 + \frac{2x}{2\sqrt{x^2 - 1}} \right) \]
\[ = \frac{1}{x + \sqrt{x^2 - 1}} \times \frac{x + \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}} = \frac{1}{x + \sqrt{x^2 - 1}} \]

- **M1**: Alt:
\[ x = \cosh y \Rightarrow \frac{dy}{dx} = \frac{1}{\sinh y} \]
\[ = \frac{1}{\sqrt{x^2 - 1}} \]

#### (iii)

\[ x = \cosh^{-1} 3 \]
\[ = \ln (3 + \sqrt{8}) \]
\[ = -\ln (3 + \sqrt{8}) \quad \text{oe} \]

- **M1**: Use of $\cosh^{-1}$
- **A1**: ft, -ve the first answer

- **A1**: ft, -ve the first answer
### Q11, (Jun 2015, Q5)

#### (i)
\[
y = \sin^{-1}(2x) \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-(2x)^2}} \cdot \frac{d(2x)}{dx} = \frac{2}{\sqrt{1-4x^2}}
\]

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#### (ii)
\[
\frac{d^2y}{dx^2} = 2 \cdot \left(-\frac{1}{2}\right)(1-4x^2)^{-\frac{3}{2}}(-8x) = \frac{8x}{(1-4x^2)^{\frac{3}{2}}}
\]
\[
= \frac{8x}{(1-4x^2)^{\frac{1}{2}}} \frac{dy}{dx} = \frac{4x}{(1-4x^2)^{\frac{1}{2}}} \frac{dy}{dx}
\]
\[
(1-4x^2) \frac{d^2y}{dx^2} = 4x \frac{dy}{dx}
\]

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<tr>
<th>B1</th>
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<td>For correct 2nd derivative</td>
<td>Using <em>their</em> ans to connect 1st and 2nd derivatives</td>
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<td>SC 2 if result obtained correctly from</td>
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<td>[ y'' = \frac{k}{\sqrt{(1-4x^2)}} ]</td>
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#### (iii)
\[
(1-4x^2) \frac{d^3y}{dx^3} - 8x \frac{d^2y}{dx^2} = 4 \frac{dy}{dx} + 4x \frac{d^2y}{dx^2}
\]
\[
(1-4x^2) \frac{d^3y}{dx^3} - 12x \frac{d^2y}{dx^2} - 4 \frac{dy}{dx} = 0
\]

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<th>A1</th>
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<td>Using result of (ii) and product rule correctly</td>
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#### (iv)
Find \(y_0, y'_0, y''_0, y'''_0\) = \{0, 2, 0, 8\}
\[
y = 0 + 2x + 0 + \frac{8x^3}{6}
\]
\[
\Rightarrow y = 2x + \frac{4x^3}{3}
\]

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<td>soi</td>
<td>Correctly substituting <em>their</em> 4 values into correct Maclaurin</td>
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