Coupled Simultaneous First Order Differential Equations (From OCR 4758)

Q1, (Jun 2007, Q4)

The following simultaneous differential equations are to be solved.

$$\frac{dx}{dt} = -5x + 4y + e^{-2t},$$
$$\frac{dy}{dt} = -9x + 7y + 3e^{-2t}.$$

(i) Show that $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = 3e^{-2t}$. [5]

(ii) Find the general solution for x in terms of t. [8]

(iii) Hence obtain the corresponding general solution for y, simplifying your answer. [4]

(iv) Given that $x = y = 0$ when $t = 0$, find the particular solutions. Find the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ when $t = 0$. Sketch graphs of the solutions. [7]

Q2, (Jun 2008, Q4)

The simultaneous differential equations

$$\frac{dx}{dt} = 4x - 6y - 9\sin t,$$
$$\frac{dy}{dt} = 3x - 5y - 7\sin t,$$

are to be solved.

(i) Show that $\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = -9\cos t - 3\sin t$. [6]

(ii) Find the general solution for x. [9]

(iii) Hence find the corresponding general solution for y. [3]

It is given that x is bounded as $t \to \infty$.

(iv) Show that y is also bounded as $t \to \infty$. [2]

(v) Given also that $y = 0$ when $t = 0$, find the particular solutions for x and y. Write down the expressions for x and y as $t \to \infty$. [4]

Q3, (Jan 2011, Q4)

The populations of foxes, x, and rabbits, y, on an island at time t are modelled by the simultaneous differential equations

$$\frac{dx}{dt} = 0.1x + 0.1y,$$
$$\frac{dy}{dt} = -0.2x + 0.3y.$$

(i) Show that $\frac{d^2x}{dt^2} - 0.4\frac{dx}{dt} + 0.05x = 0.$ **[5]**

(ii) Find the general solution for x. **[4]**

(iii) Find the corresponding general solution for y. **[4]**

Initially there are x_0 foxes and y_0 rabbits.

(iv) Find the particular solutions. **[4]**

(v) In the case $y_0 = 10x_0$, find the time at which the model predicts the rabbits will die out. Determine whether the model predicts the foxes die out before the rabbits. **[7]**

Q4, (Jan 2012, Q4)

The simultaneous differential equations

$$\frac{dx}{dt} = -x + 2y$$
$$\frac{dy}{dt} = -x - 4y + e^{-2t}$$

are to be solved.

(i) Eliminate y to obtain a second order differential equation for x in terms of t. Hence find the general solution for x. **[14]**

(ii) Find the corresponding general solution for y. **[3]**

Initially $x = 5$ and $y = 0$.

(iii) Find the particular solutions. **[4]**

(iv) Show that $\frac{y}{x} \to -\frac{1}{2}$ as $t \to \infty$. Show also that there is no value of t for which $\frac{y}{x} = -\frac{1}{2}$. **[3]**

Q5, (Jan 2013, Q4)

The simultaneous differential equations

$$\frac{dx}{dt} = -\frac{1}{2}x - \frac{3}{2}y + t$$
$$\frac{dy}{dt} = \frac{3}{2}x - \frac{1}{2}y + 2t$$

are to be solved.

(i) Eliminate y to obtain a second order differential equation for x in terms of t. Hence find the general solution for x. **[13]**

(ii) Find the corresponding general solution for y. **[4]**

When $t = 0$, $x = 1$ and $y = 0$.

(iii) Find the particular solutions. **[3]**

(iv) Show that in this case $x + y$ tends to a finite limit as $t \to \infty$ and state its value. Determine whether $x + y$ is equal to this limit for any values of t. **[4]**

Q6, (Jun 2015, Q4)

Two species of small rodent, X and Y, compete for survival in the same environment. The populations of the species, at time t years, are x and y respectively and they are modelled by the simultaneous differential equations

$$\frac{dx}{dt} = 2(x - y),$$
$$\frac{dy}{dt} = \frac{3}{8}\left(x - 80e^{-\frac{1}{2}t}\right).$$

(i) Show that

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + \frac{3}{4}x = 60e^{-\frac{1}{2}t}.$$

Find the general solution for x. **[10]**

(ii) Find the corresponding general solution for y. **[3]**

When $t = 0$, $x = 40$ and $y = 50$.

(iii) Find the particular solutions for x and y. **[4]**

(iv) Find the time T at which the model predicts that the rodents of species X will die out. Find the population of species Y predicted at this time. **[6]**

(v) Comment on the suitability of the model for times greater than T. **[1]**

Q7, (Jun 2017, Q4)

Two species of insects, X and Y, compete for survival on an island. The populations of the species are x and y respectively at time t, where t is measured in tens of years. The situation is modelled by the simultaneous differential equations

$$\frac{dx}{dt} = 2x + 2y,$$

$$\frac{dy}{dt} = 6y - 4x.$$

(i) Eliminate y to obtain a second order differential equation for x in terms of t. Hence find the general solution for x. **[7]**

(ii) Find the corresponding general solution for y. **[4]**

When $t = 0$, $\frac{dx}{dt} = 10$ and the population of species Y is k times the population of species X, where k is a positive constant.

(iii) Find the particular solutions for x and y, in terms of t and k. **[5]**

Consider the case $k = 6$.

(iv) Determine whether the model predicts that species X or species Y dies out first. State the value of t at which this first species dies out. **[7]**

(v) Comment on why the time predicted by the model for the second species to die out is unreliable. **[1]**

Q8, (Jun 2018, Q4)

The simultaneous differential equations

$$\frac{dx}{dt} = 7x + 2y + 13e^{4t},$$

$$\frac{dy}{dt} = -9x + y + e^{7t}$$

are to be solved.

(i) Eliminate x to obtain a second order differential equation for y in terms of t. Hence find the general solution for y. **[12]**

(ii) Given that $y = -3$ and $\frac{dy}{dt} = 60$ when $t = 0$, find the particular solution for y. **[4]**

(iii) Find the corresponding particular solution for x. **[2]**

(iv) Find the smallest positive value of t for which $y = 0$. **[4]**

(v) Show that $\frac{y}{x} \to 0$ as $t \to \infty$. **[2]**
