## Centres of Mass Involving Integration (From OCR 4763)

**Q1, (Jan 2006, Q4)**

The region between the curve $y = 4 - x^2$ and the $x$-axis, from $x = 0$ to $x = 2$, is occupied by a uniform lamina. The units of the axes are metres.

**(i)** Show that the coordinates of the centre of mass of this lamina are $(0.75, 1.6)$. [9]

This lamina and another exactly like it are attached to a uniform rod PQ, of mass 12 kg and length 8 m, to form a rigid body as shown in Fig. 4. Each lamina has mass 6.5 kg. The ends of the rod are at P$(-4, 0)$ and Q$(4, 0)$. The rigid body lies entirely in the $(x, y)$ plane.

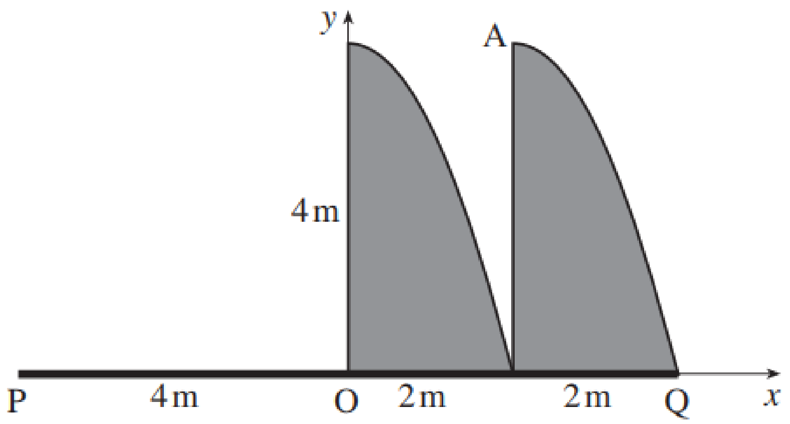


**Fig. 4**

**(ii)** Find the coordinates of the centre of mass of the rigid body. [5]

The rigid body is freely suspended from the point A$(2, 4)$ and hangs in equilibrium.

**(iii)** Find the angle that PQ makes with the horizontal. [4]

**Q2, (June 2006, Q4)**

The region bounded by the curve $y = \sqrt{x}$, the $x$-axis and the lines $x = 1$ and $x = 4$ is rotated through $2\pi$ radians about the $x$-axis to form a uniform solid of revolution.

**(i)** Find the $x$-coordinate of the centre of mass of this solid. [6]

From this solid, the cylinder with radius 1 and length 3 with its axis along the $x$-axis (from $x = 1$ to $x = 4$) is removed.

**(ii)** Show that the centre of mass of the remaining object, Q, has $x$-coordinate 3. [5]

This object Q has weight 96 N and it is supported, with its axis of symmetry horizontal, by a string passing through the cylindrical hole and attached to fixed points A and B (see Fig. 4). AB is horizontal and the sections of the string attached to A and B are vertical. There is sufficient friction to prevent slipping.

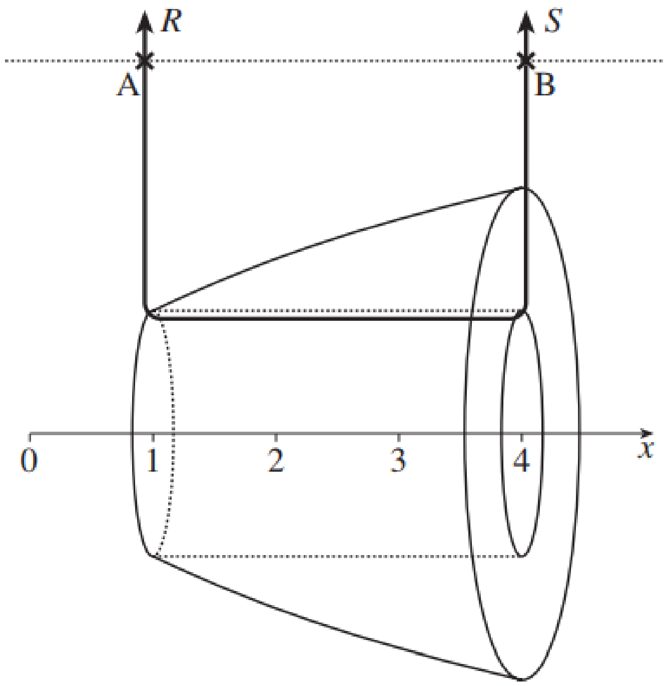


**Fig. 4**

**(iii)** Find the support forces, $R$ and $S$, acting on the string at A and B

($A$) when the string is light, [4]

($B$) when the string is heavy and uniform with a total weight of 6 N. [3]

**Q3, (Jan 2007, Q4)**

In this question, $a$ is a constant with $a > 1$.

Fig. 4 shows the region bounded by the curve $y = \frac{1}{x^2}$ for $1 \leqslant x \leqslant a$, the $x$-axis, and the lines $x = 1$ and $x = a$.

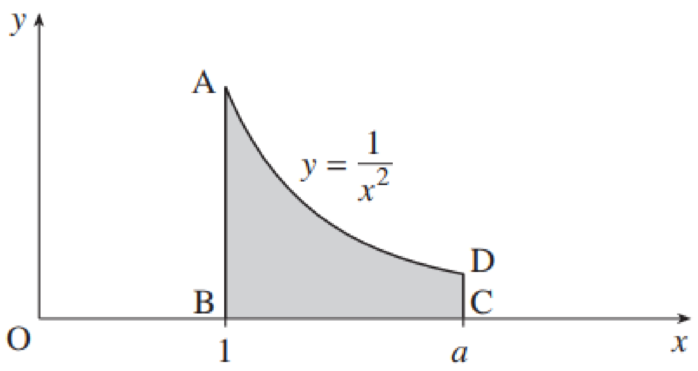


**Fig. 4**

This region is occupied by a uniform lamina ABCD, where A is $(1, 1)$, B is $(1, 0)$, C is $(a, 0)$ and D is $\left(a, \frac{1}{a^2}\right)$. The centre of mass of this lamina is $(\bar{x}, \bar{y})$.

**(i)** Find $\bar{x}$ in terms of $a$, and show that $\bar{y} = \frac{a^3 - 1}{6(a^3 - a^2)}$. [8]

**(ii)** In the case $a = 2$, the lamina is freely suspended from the point A, and hangs in equilibrium. Find the angle which AB makes with the vertical. [3]

The region shown in Fig. 4 is now rotated through $2\pi$ radians about the $x$-axis to form a uniform solid of revolution.

**(iii)** Find the $x$-coordinate of the centre of mass of this solid of revolution, in terms of $a$, and show that it is less than 1.5. [7]

**Q4, (Jun 2007, Q4)**

**(a)** The region bounded by the curve $y = x^3$ for $0 \leqslant x \leqslant 2$, the $x$-axis and the line $x = 2$, is occupied by a uniform lamina. Find the coordinates of the centre of mass of this lamina. [8]

**(b)** The region bounded by the circular arc $y = \sqrt{4 - x^2}$ for $1 \leqslant x \leqslant 2$, the $x$-axis and the line $x = 1$, is rotated through $2\pi$ radians about the $x$-axis to form a uniform solid of revolution, as shown in Fig. 4.1.

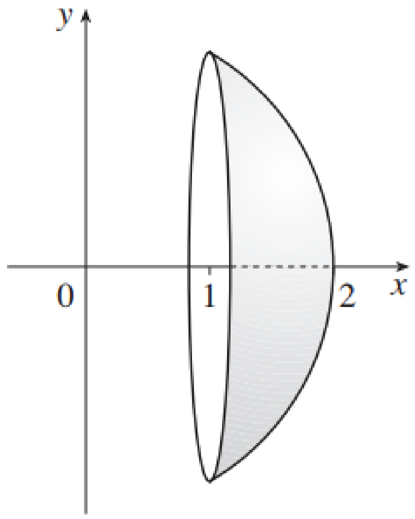


**Fig. 4.1**

**(i)** Show that the $x$-coordinate of the centre of mass of this solid of revolution is 1.35. [6]

This solid is placed on a rough horizontal surface, with its flat face in a vertical plane. It is held in equilibrium by a light horizontal string attached to its highest point and perpendicular to its flat face, as shown in Fig. 4.2.

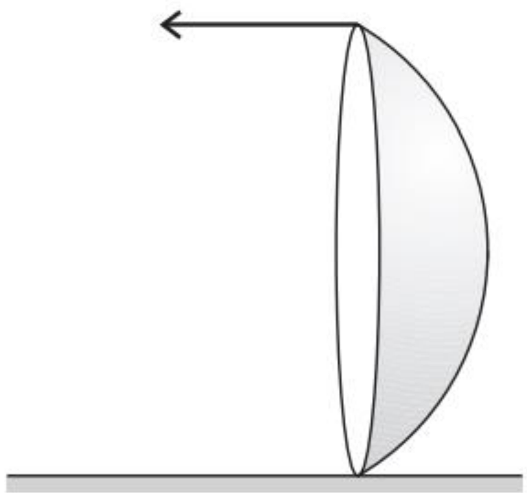

**Fig. 4.2**

**(ii)** Find the least possible coefficient of friction between the solid and the horizontal surface. [4]

**Q5, (Jan 2008, Q4)**

Fig. 4.1 shows the region $R$ bounded by the curve $y = x^{-\frac{1}{3}}$ for $1 \leqslant x \leqslant 8$, the $x$-axis, and the lines $x = 1$ and $x = 8$.

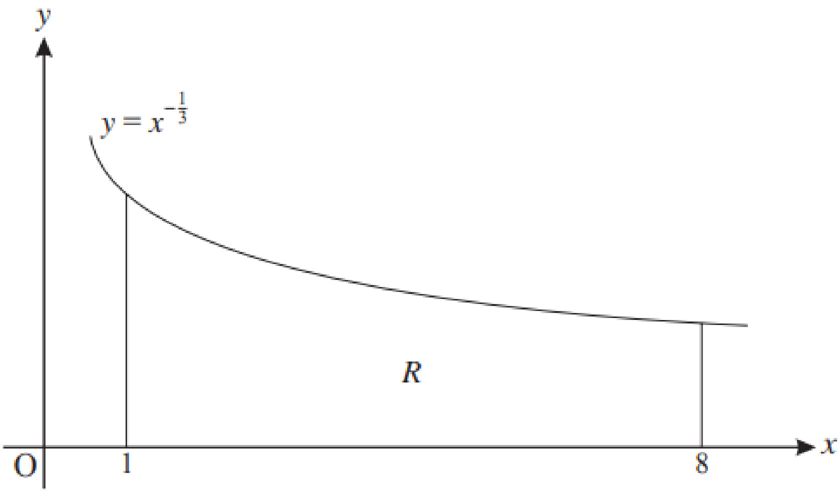


**Fig. 4.1**

**(i)** Find the $x$-coordinate of the centre of mass of a uniform solid of revolution obtained by rotating $R$ through $2\pi$ radians about the $x$-axis. [6]

**(ii)** Find the coordinates of the centre of mass of a uniform lamina in the shape of the region $R$. [8]

**(iii)** Using your answer to part **(ii)**, or otherwise, find the coordinates of the centre of mass of a uniform lamina in the shape of the region (shown shaded in Fig. 4.2) bounded by the curve $y = x^{-\frac{1}{3}}$ for $1 \leqslant x \leqslant 8$, the line $y = \frac{1}{2}$ and the line $x = 1$. [4]

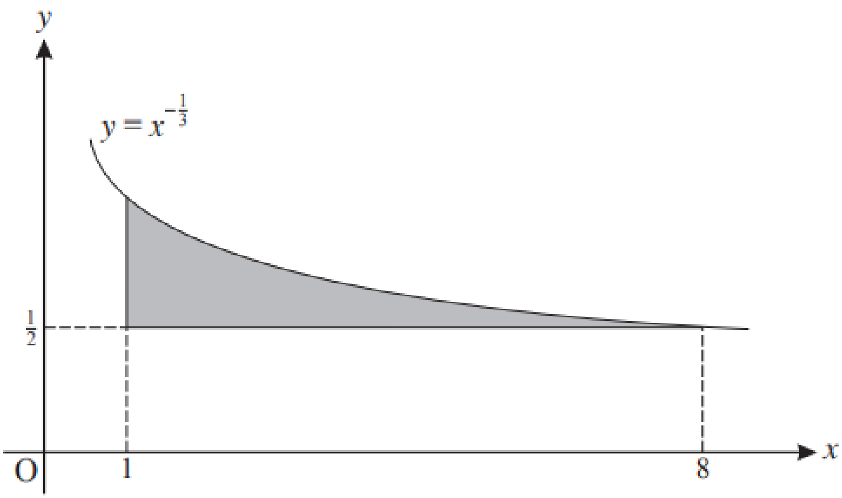


**Fig. 4.2**

**Q6, (Jan 2009, Q4)**

**(a)** The region bounded by the $x$-axis and the semicircle $y = \sqrt{a^2 - x^2}$ for $-a \leqslant x \leqslant a$ is occupied by a uniform lamina with area $\frac{1}{2}\pi a^2$. Show by integration that the $y$-coordinate of the centre of mass of this lamina is $\dfrac{4a}{3\pi}$. **[4]**

**(b)** A uniform solid cone is formed by rotating the region between the $x$-axis and the line $y = mx$, for $0 \leqslant x \leqslant h$, through $2\pi$ radians about the $x$-axis.

**(i)** Show that the $x$-coordinate of the centre of mass of this cone is $\frac{3}{4}h$. **[6]**

[You may use the formula $\frac{1}{3}\pi r^2 h$ for the volume of a cone.]

From such a uniform solid cone with radius 0.7 m and height 2.4 m, a cone of material is removed. The cone removed has radius 0.4 m and height 1.1 m; the centre of its base coincides with the centre of the base of the original cone, and its axis of symmetry is also the axis of symmetry of the original cone. Fig. 4 shows the resulting object; the vertex of the original cone is V, and A is a point on the circumference of its base.

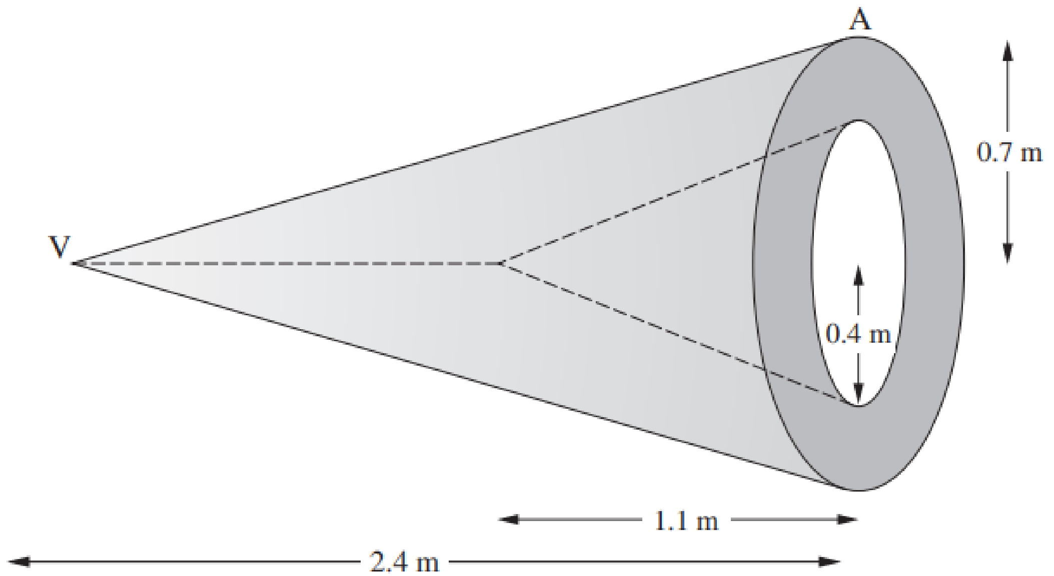


**Fig. 4**

**(ii)** Find the distance of the centre of mass of this object from V. **[5]**

This object is suspended by a string attached to a point Q on the line VA, and hangs in equilibrium with VA horizontal.

**(iii)** Find the distance VQ. **[3]**

**Q7, (Jun 2010, Q3)**

In this question, give your answers in an exact form.

The region $R_1$ (shown in Fig. 3) is bounded by the $x$-axis, the lines $x = 1$ and $x = 5$, and the curve $y = \frac{1}{x}$ for $1 \leqslant x \leqslant 5$.

**(i)** A uniform solid of revolution is formed by rotating the region $R_1$ through $2\pi$ radians about the $x$-axis. Find the $x$-coordinate of the centre of mass of this solid. **[5]**

**(ii)** Find the coordinates of the centre of mass of a uniform lamina occupying the region $R_1$. **[7]**

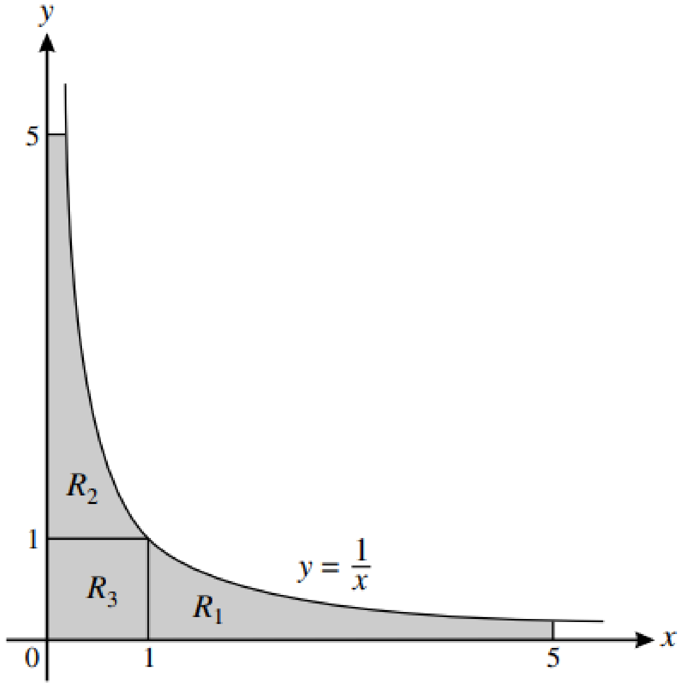


**Fig. 3**

The region $R_2$ is bounded by the $y$-axis, the lines $y = 1$ and $y = 5$, and the curve $y = \frac{1}{x}$ for $\frac{1}{5} \leqslant x \leqslant 1$.
The region $R_3$ is the square with vertices (0, 0), (1, 0), (1, 1) and (0, 1).

**(iii)** Write down the coordinates of the centre of mass of a uniform lamina occupying the region $R_2$. **[2]**

**(iv)** Find the coordinates of the centre of mass of a uniform lamina occupying the region consisting of $R_1$, $R_2$ and $R_3$ (shown shaded in Fig. 3). **[4]**

Q8, (Jun 2013, Q4)

**(a)** A uniform solid of revolution $S$ is formed by rotating the region enclosed between the $x$-axis and the curve $y = x\sqrt{4-x}$ for $0 \leqslant x \leqslant 4$ through $2\pi$ radians about the $x$-axis, as shown in Fig. 4.1. O is the origin and the end A of the solid is at the point $(4, 0)$.

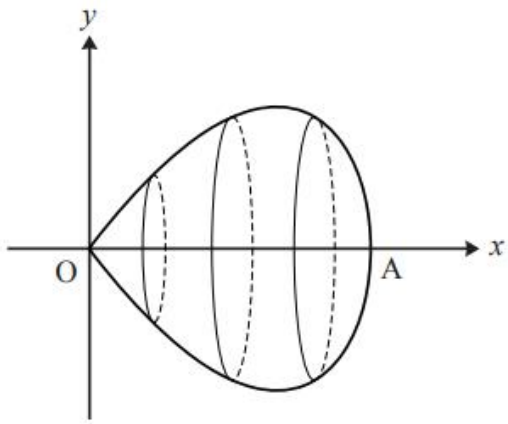


**Fig. 4.1**

**(i)** Find the $x$-coordinate of the centre of mass of the solid $S$. **[6]**

The solid $S$ has weight $W$, and it is freely hinged to a fixed point at O. A horizontal force, of magnitude $W$ acting in the vertical plane containing OA, is applied at the point A, as shown in Fig. 4.2. $S$ is in equilibrium.

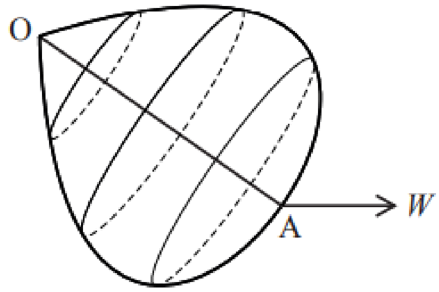


**Fig. 4.2**

**(ii)** Find the angle that OA makes with the vertical. **[3]**

**(b)** Fig. 4.3 shows the region bounded by the $x$-axis, the $y$-axis, the line $y = 8$ and the curve $y = (x-2)^3$ for $0 \leqslant y \leqslant 8$.

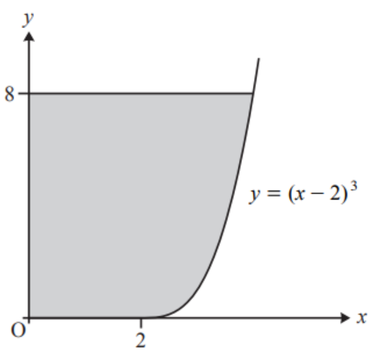


**Fig. 4.3**

Find the coordinates of the centre of mass of a uniform lamina occupying this region. **[9]**

**Q9, (Jun 2014, Q4)**

The region $R$ is bounded by the $x$-axis, the $y$-axis, the curve $y = e^{-x}$ and the line $x = k$, where $k$ is a positive constant.

**(i)** The region $R$ is rotated through $2\pi$ radians about the $x$-axis to form a uniform solid of revolution. Find the $x$-coordinate of the centre of mass of this solid, and show that it can be written in the form

$$\frac{1}{2} - \frac{k}{e^{2k} - 1}.$$ **[7]**

**(ii)** The solid in part **(i)** is placed with its larger circular face in contact with a rough plane inclined at 60° to the horizontal, as shown in Fig. 4, and you are given that no slipping occurs.

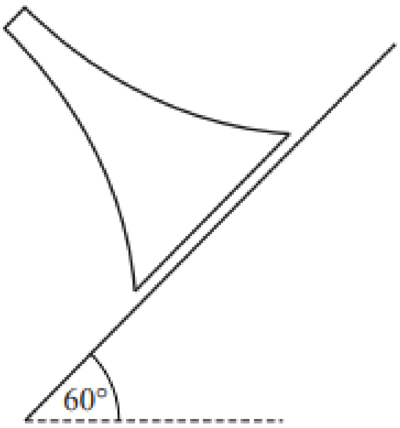


**Fig. 4**

Show that, whatever the value of $k$, the solid will not topple. **[4]**

**(iii)** A uniform lamina occupies the region $R$. Find, in terms of $k$, the coordinates of the centre of mass of this lamina. **[7]**

**Q10, (Jun 2015, Q4)**

**(a)** A uniform lamina occupies the region bounded by the $x$-axis and the curve $y = \dfrac{x^2(a-x)}{a^2}$ for $0 \leqslant x \leqslant a$. Find the coordinates of the centre of mass of this lamina. **[9]**

**(b)** The region $A$ is bounded by the $x$-axis, the $y$-axis, the curve $y = \sqrt{x^2 + 16}$ and the line $x = 3$. The region $B$ is bounded by the $y$-axis, the curve $y = \sqrt{x^2 + 16}$ and the line $y = 5$. These regions are shown in Fig. 4.

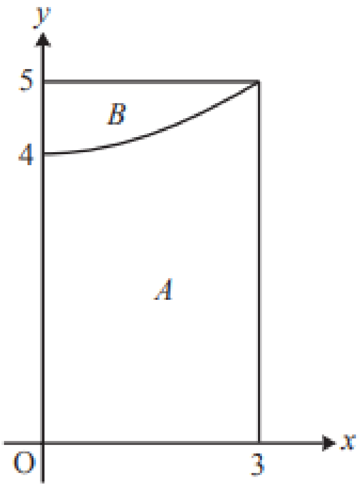


**Fig. 4**

**(i)** Find the $x$-coordinate of the centre of mass of the uniform solid of revolution formed when the region $A$ is rotated through $2\pi$ radians about the $x$-axis. **[5]**

**(ii)** Using your answer to part **(i)**, or otherwise, find the $x$-coordinate of the centre of mass of the uniform solid of revolution formed when the region $B$ is rotated through $2\pi$ radians about the $x$-axis. **[4]**