Proof By Induction (Divisibility) Exam Questions (From OCR 4725 unless otherwise stated)

Q1, (Jan 2007, Q6)

(i)	B1		Correct expression for u_{n+1}
	M1		Attempt to expand and simplify
$u_{n+1} - u_n = 2n + 4$	A1	3	Obtain given answer correctly
(ii)	B1		State $u_1 = 4$ (or $u_2 = 10$)and is divisible by 2
	M1		State induction hypothesis true for u_n
	M1		Attempt to use result in (ii)
	A1		Correct conclusion reached for u_{n+1}
	A1	5	Clear,explicit statement of induction conclusion
		8	

Q2, (Jan 2009, Q7)

(i) $13^n + 6^{n-1} + 13^{n+1} + 6^n$	B1		Correct expression seen
	M1		Attempt to factorise both terms in (i)
	A1	3	Obtain correct expression
(ii)	B1		Check that result is true for $n = 1$ (or 2)
	B1		Recognise that (i) is divisible by 7
	B1		Deduce that u_{n+1} is divisible by 7
	B1	4	Clear statement of Induction conclusion
		7	

Q3, (Jun 2014, Q10)

(i)		6 27 129	B1 B1 **[2]**	Obtain correct values Obtain 3rd correct value
(ii)		3	B1ft **[1]**	State a correct value
(iii)		$5^{n+1} + 2^n$	B1 M1 A1 A1 B1 **[5]**	Correct expression for u_{n+1} seen Attempt to factorise $u_{n+1} + u_n$ Obtain correct simplified answer Clear explanation why u_{n+1} is divisible by 3 Clear statement of induction process

Q4, (Edexcel 6667, Jun 2009, Q80

(a)	$f(1) = 5 + 8 + 3 = 16$, (which is divisible by 4). ($\therefore$ True for $n = 1$).	B1
	Using the formula to write down f(k + 1), $f(k+1) = 5^{k+1} + 8(k+1) + 3$	M1 A1
	$f(k+1) - f(k) = 5^{k+1} + 8(k+1) + 3 - 5^k - 8k - 3$	M1
	$= 5(5^k) + 8k + 8 + 3 - 5^k - 8k - 3 = 4(5^k) + 8$	A1
	$f(k+1) = 4(5^k + 2) + f(k)$, **which is divisible by** 4	A1ft
	$\therefore$ True for $n = k + 1$ **if** true for $n = k$. True for $n = 1$, $\therefore$ true for all n.	A1cso (7)

Q5, (Edexcel 6667, Jun 2010, Q7)

(a) LHS = $f(k+1) = 2^{k+1} + 6^{k+1}$	OR RHS =	M1
	$= 6f(k) - 4(2^k) = 6(2^k + 6^k) - 4(2^k)$	
$= 2(2^k) + 6(6^k)$	$= 2(2^k) + 6(6^k)$	A1
$= 6(2^k + 6^k) - 4(2^k) = 6f(k) - 4(2^k)$	$= 2^{k+1} + 6^{k+1} = f(k+1)$ (*)	A1 (3)
OR $f(k+1) - 6f(k) = 2^{k+1} + 6^{k+1} - 6(2^k + 6^k)$		M1
$= (2-6)(2^k) = -4.2^k$, and so $f(k+1) = 6f(k) - 4(2^k)$		A1, A1 (3)
(b) $n = 1$: $f(1) = 2^1 + 6^1 = 8$, **which is divisible** by 8		B1
Either Assume f(k) divisible by 8 and try to use f(k + 1) = 6f(k) - 4(2^k)	**Or** Assume f(k) divisible by 8 and try to use f (k + 1)– f (k) or f (k + 1) + f (k) including factorising $6^k = 2^k 3^k$	M1
Show $4(2^k) = 4 \times 2(2^{k-1}) = 8(2^{k-1})$ or $8(\frac{1}{2}2^k)$ Or valid statement	$= 2^3 2^{k-3}(1 + 5.3^k)$ or $= 2^3 2^{k-3}(3 + 7.3^k)$ o.e.	A1
Deduction that result is implied for $n = k + 1$ and so is true for positive integers by induction (may include $n = 1$ true here)	Deduction that result is implied for $n = k + 1$ and so is true for positive integers by induction (must include explanation of why $n = 2$ is also true here)	A1cso (4) **7 marks**

Q6, (Edexcel 6667, Jun 2012, Q10)

$\mathrm{f}(n) = 2^{2n-1} + 3^{2n-1}$ is divisible by 5.		
$\mathrm{f}(1) = 2^1 + 3^1 = 5$,	Shows that $\mathrm{f}(1) = 5$.	B1
Assume that for $n = k$, $\mathrm{f}(k) = 2^{2k-1} + 3^{2k-1}$ is divisible by 5 for $k \in ¢^+$.		
$\mathrm{f}(k+1) - \mathrm{f}(k) = 2^{2(k+1)-1} + 3^{2(k+1)-1} - \left(2^{2k-1} + 3^{2k-1}\right)$	M1: Attempts $\mathrm{f}(k+1) - \mathrm{f}(k)$. A1: Correct expression for f(k + 1) (Can be unsimplified)	M1A1
$= 2^{2k+1} + 3^{2k+1} - 2^{2k-1} - 3^{2k-1}$		
$= 2^{2k-1+2} + 3^{2k-1+2} - 2^{2k-1} - 3^{2k-1}$		
$= 4\left(2^{2k-1}\right) + 9\left(3^{2k-1}\right) - 2^{2k-1} - 3^{2k-1}$	Achieves an expression in 2^{2k-1} and 3^{2k-1}	M1
$= 3\left(2^{2k-1}\right) + 8\left(3^{2k-1}\right)$		
$= 3\left(2^{2k-1}\right) + 3\left(3^{2k-1}\right) + 5\left(3^{2k-1}\right)$		
$= 3\mathrm{f}(k) + 5\left(3^{2k-1}\right)$		
$\therefore \mathrm{f}(k+1) = 4\mathrm{f}(k) + 5\left(3^{2k-1}\right)$ or $4(2^{2k-1} + 3^{2k-1}) + 5\left(3^{2k-1}\right)$	Where f(k + 1) is correct and is clearly a multiple of 5.	A1
If the result is true for $n = k$, then it is now true for $n = k+1$. As the result has shown to be true for $n = 1$, then the result is true for all n.	Correct conclusion **at the end,** at least as given, and all previous marks scored.	A1 cso
		[6]
		6 marks

Q7, (Edexcel 6667A, Jan 2014, Q10)

(i)	$u_{n+1} = 5u_n + 3$, $u_1 = 3$ and $u_n = \frac{3}{4}(5^n - 1)$		
	$n = 1$; $u_1 = \frac{3}{4}(5^1 - 1) = \frac{3}{4}(4) = 3$ So u_n is true when $n = 1$.	Check that $u_n = \frac{3}{4}(5^n - 1)$ yields 3 when $n = 1$.	B1
	Assume that for $n = k$ that, $u_k = \frac{3}{4}(5^k - 1)$ is true for $k \in \mathbb{Z}^+$. Then $u_{k+1} = 5u_k + 3$		
	$= 5\left(\frac{3}{4}(5^k - 1)\right) + 3$	Substituting $u_k = \frac{3}{4}(5^k - 1)$ into $u_{k+1} = 5u_k + 3$	M1
	$= \frac{3}{4}(5)^{k+1} - \frac{15}{4} + 3$	An attempt to multiply out in order to achieve $\pm\lambda(5^{k+1}) \pm$ constant	M1
	$= \frac{3}{4}(5)^{k+1} - \frac{3}{4}$		
	$= \frac{3}{4}(5^{k+1} - 1)$	$\frac{3}{4}(5^{k+1} - 1)$	A1
	Therefore, the general statement, $u_n = \frac{3}{4}(5^n - 1)$ is true when $n = k+1$. (As u_n is true for $n = 1$,) then u_n is true for all positive integers by mathematical induction	True when $n = k+1$, then by induction the result is true for all positive integers.	A1
			[5]
(ii)	$f(n) = 5(5^n) - 4n - 5$ is divisible by 16		
	$f(1) = 5(5^1) - 4(1) - 5 = 16$, {which is divisible by 16}. $\{\therefore f(n)$ is divisible by 16 when $n = 1.\}$	Shows that $f(1) = 16$	B1
	Assume that for $n = k$, $f(k) = 5(5^k) - 4k - 5$ is divisible by 16 for $k \in \mathbb{Z}^+$.		
	$f(k+1) - f(k) = 5(5^{k+1}) - 4(k+1) - 5 - \left(5(5^k) - 4k - 5\right)$	Applies $f(k+1) - f(k)$. Correct expression for $f(k+1) - f(k)$.	M1 A1
	$= 5(5^{k+1}) - 4k - 4 - 5 - 5(5^k) + 4k + 5$		
	$= 25(5^k) - 4k - 4 - 5 - 5(5^k) + 4k + 5$	Achieves an expression in 5^k.	M1
	$= 20(5^k) - 4$ $= 4\left(5(5^k) - 4k - 5\right) + 16k + 20 - 4$ $= 4\left(5(5^k) - 4k - 5\right) + 16k + 16$ $= 4f(k) + 16(k+1)$		
	$\therefore f(k+1) = 5f(k) + 16(k+1)$ $\{\therefore f(k+1) = 5f(k) + 16(k+1)$, which is divisible by 16 as both $5f(k)$ and $16(k+1)$ are both divisible by 16.}	$f(k+1) = 5f(k) + 16(k+1)$	A1
	If the result is true for $n = k$, then it is now true for $n = k+1$. As the result has shown to be true for $n = 1$, then the result is true for all n.	Correct conclusion	A1 cso
			[6] **11**

Q8, (Edexcel 6667, Jun 2014, Q9)

$f(n) = 8^n - 2^n$ is divisible by 6.		
$f(1) = 8^1 - 2^1 = 6$,	Shows that $f(1) = 6$	B1
Assume that for $n = k$, $f(k) = 8^k - 2^k$ is divisible by 6.		
$f(k+1) - f(k) = 8^{k+1} - 2^{k+1} - \left(8^k - 2^k\right)$	Attempt $f(k+1) - f(k)$	M1
$= 8^k(8-1) + 2^k(1-2) = 7\times 8^k - 2^k$		
$= 6\times 8^k + 8^k - 2^k \left(= 6\times 8^k + f(k)\right)$	M1: Attempt $f(k+1) - f(k)$ as a multiple of 6 A1: rhs a correct multiple of 6	M1A1
$f(k+1) = 6\times 8^k + 2f(k)$	Completes to $f(k+1) =$ a multiple of 6	A1
If the result is **true for** $n = k$, then it is now **true for** $n = k+1$. As the result has been shown to be **true for** $n = 1$, then the result is **true for all** n ($\in \mathbb{Z}^+$.)		A1cso
	Do not award final A if n defined incorrctly e.g. 'n is an integer' award A0	
		(6)
		Total 6