STEP II – Hooke's Law

Hooke's law

Know, understand and use Hooke's law for strings and springs, including the formula $T = kx = \frac{\lambda x}{l}$ where k is the stiffness and λ is the modulus of elasticity.

Understand and use elastic potential energy, including knowledge of the formula $E = \frac{1}{2}kx^2 = \frac{\lambda x^2}{2l}$.

Q1, (STEP III, 2014, Q10)

Two particles X and Y, of equal mass m, lie on a smooth horizontal table and are connected by a light elastic spring of natural length a and modulus of elasticity λ. Two more springs, identical to the first, connect X to a point P on the table and Y to a point Q on the table. The distance between P and Q is $3a$.

Initially, the particles are held so that $XP = a$, $YQ = \frac{1}{2}a$, and $PXYQ$ is a straight line. The particles are then released.

At time t, the particle X is a distance $a + x$ from P and the particle Y is a distance $a + y$ from Q. Show that

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{\lambda}{a}(2x + y)$$

and find a similar expression involving $\frac{\mathrm{d}^2y}{\mathrm{d}t^2}$. Deduce that

$$x - y = A\cos\omega t + B\sin\omega t$$

where A and B are constants to be determined and $ma\omega^2 = \lambda$. Find a similar expression for $x + y$.

Show that Y will never return to its initial position.

Q2, (STEP III, 2017, Q9)

Two particles A and B of masses m and $2m$, respectively, are connected by a light spring of natural length a and modulus of elasticity λ. They are placed on a smooth horizontal table with AB perpendicular to the edge of the table, and A is held on the edge of the table. Initially the spring is at its natural length.

Particle A is released. At a time t later, particle A has dropped a distance y and particle B has moved a distance x from its initial position (where $x < a$). Show that $y + 2x = \frac{1}{2}gt^2$.

The value of λ is such that particle B reaches the edge of the table at a time T given by $T = \sqrt{6a/g}$. By considering the total energy of the system (without solving any differential equations), show that the speed of particle B at this time is $\sqrt{2ag/3}$.

Q3. (STEP III, 2015, Q9)

A particle P of mass m moves on a smooth fixed straight horizontal rail and is attached to a fixed peg Q by a light elastic string of natural length a and modulus λ. The peg Q is a distance a from the rail. Initially P is at rest with $PQ = a$.

An impulse imparts to P a speed v along the rail. Let x be the displacement at time t of P from its initial position. Obtain the equation

$$\dot{x}^2 = v^2 - k^2\left(\sqrt{x^2+a^2} - a\right)^2$$

where $k^2 = \lambda/(ma)$, $k > 0$ and the dot denotes differentiation with respect to t.

Find, in terms of k, a and v, the greatest value, x_0, attained by x. Find also the acceleration of P at $x = x_0$.

Obtain, in the form of an integral, an expression for the period of the motion. Show that, in the case $v \ll ka$ (that is, v is much less than ka), this is approximately

$$\sqrt{\frac{32a}{kv}}\int_0^1 \frac{1}{\sqrt{1-u^4}}\,\mathrm{d}u\,.$$

Q4, (STEP III, 2016, Q9)

Three pegs P, Q and R are fixed on a smooth horizontal table in such a way that they form the vertices of an equilateral triangle of side $2a$. A particle X of mass m lies on the table. It is attached to the pegs by three springs, PX, QX and RX, each of modulus of elasticity λ and natural length l, where $l < \frac{2}{\sqrt{3}}a$. Initially the particle is in equilibrium. Show that the extension in each spring is $\frac{2}{\sqrt{3}}a - l$.

The particle is then pulled a small distance directly towards P and released. Show that the tension T in the spring RX is given by

$$T = \frac{\lambda}{l}\left(\sqrt{\frac{4a^2}{3} + \frac{2ax}{\sqrt{3}} + x^2}\; - l\right),$$

where x is the displacement of X from its equilibrium position.

Show further that the particle performs approximate simple harmonic motion with period

$$2\pi\sqrt{\frac{4mla}{3(4a - \sqrt{3}\,l)\lambda}}\,.$$

Q5, (STEP III, 2011, Q10)

Particles P and Q, each of mass m, lie initially at rest a distance a apart on a smooth horizontal plane. They are connected by a light elastic string of natural length a and modulus of elasticity $\frac{1}{2}ma\omega^2$, where ω is a constant.

Then P receives an impulse which gives it a velocity u directly away from Q. Show that when the string next returns to length a, the particles have travelled a distance $\frac{1}{2}\pi u/\omega$, and find the speed of each particle.

Find also the total time between the impulse and the subsequent collision of the particles.
