STEP III - Second Order Differential Equations

STEP III Specification

Differential equations

Solve differential equations of the form $y'' + ay' + by = 0$, where a and b are constants, by using the auxiliary equation.

Know, understand and use the form of the solution of the differential equations in cases when the discriminant of the auxiliary equation is positive, zero, or negative.

Solve differential equations of the form $y'' + ay' + by = \mathrm{f}(x)$ where a and b are constants, by solving the homogeneous case and adding a particular integral to the complementary function (in cases where $\mathrm{f}(x)$ is a polynomial, exponential, or trigonometric function).

Solve the equation for simple harmonic motion $\ddot{x} = -\omega^2 x$ and relate the solution to the motion, ***and understand the implications in physical situations.***

Model damped oscillations using second order differential equations and interpret their solutions.

Analyse models of situations with one independent variable and two dependent variables as a pair of coupled first order simultaneous equations and be able to solve them, for example predator-prey models.

Use given substitutions to transform differential equations.

Q1, (STEP III, 2011, Q2)

(i) Find the general solution of the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \left(\frac{x+2}{x+1}\right)u = 0\,.$$

(ii) Show that substituting $y = z\mathrm{e}^{-x}$ (where z is a function of x) into the second order differential equation

$$(x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0 \qquad (*)$$

leads to a first order differential equation for $\frac{\mathrm{d}z}{\mathrm{d}x}$. Find z and hence show that the general solution of $(*)$ is

$$y = Ax + B\mathrm{e}^{-x}\,,$$

where A and B are arbitrary constants.

(iii) Find the general solution of the differential equation

$$(x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - y = (x+1)^2.$$

Q2, (STEP III, 2012, Q1)

Given that $z = y^n \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$, show that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = y^{n-1}\frac{\mathrm{d}y}{\mathrm{d}x}\left(n\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right).$$

(i) Use the above result to show that the solution to the equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sqrt{y} \qquad (y > 0)$$

that satisfies $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$ is $y = \left(\frac{3}{8}x^2 + 1\right)^{\frac{2}{3}}$.

(ii) Find the solution to the equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y^2 = 0$$

that satisfies $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$.

Q6, (STEP III, 2007, Q8)

(i) Find functions $\mathrm{a}(x)$ and $\mathrm{b}(x)$ such that $u = x$ and $u = \mathrm{e}^{-x}$ both satisfy the equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \mathrm{a}(x)\frac{\mathrm{d}u}{\mathrm{d}x} + \mathrm{b}(x)u = 0\,.$$

For these functions $\mathrm{a}(x)$ and $\mathrm{b}(x)$, write down the general solution of the equation.

Show that the substitution $y = \frac{1}{3u}\frac{\mathrm{d}u}{\mathrm{d}x}$ transforms the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 3y^2 + \frac{x}{1+x}y = \frac{1}{3(1+x)} \qquad (*)$$

into

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{x}{1+x}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{1}{1+x}u = 0$$

and hence show that the solution of equation $(*)$ that satisfies $y = 0$ at $x = 0$ is given by $y = \frac{1 - \mathrm{e}^{-x}}{3(x + \mathrm{e}^{-x})}$.

(ii) Find the solution of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + \frac{x}{1-x}y = \frac{1}{1-x}$$

that satisfies $y = 2$ at $x = 0$.

Q3, (STEP III, 2012, Q7)

A pain-killing drug is injected into the bloodstream. It then diffuses into the brain, where it is absorbed. The quantities at time t of the drug in the blood and the brain respectively are $y(t)$ and $z(t)$. These satisfy

$$\dot{y} = -2(y - z)\,, \qquad \dot{z} = -\dot{y} - 3z\,,$$

where the dot denotes differentiation with respect to t.

Obtain a second order differential equation for y and hence derive the solution

$$y = A\mathrm{e}^{-t} + B\mathrm{e}^{-6t}\,, \qquad z = \tfrac{1}{2}A\mathrm{e}^{-t} - 2B\mathrm{e}^{-6t}\,,$$

where A and B are arbitrary constants.

(i) Obtain the solution that satisfies $z(0) = 0$ and $y(0) = 5$. The quantity of the drug in the brain for this solution is denoted by $z_1(t)$.

(ii) Obtain the solution that satisfies $z(0) = z(1) = c$, where c is a given constant. The quantity of the drug in the brain for this solution is denoted by $z_2(t)$.

(iii) Show that for $0 \leqslant t \leqslant 1$,

$$z_2(t) = \sum_{n=-\infty}^{0} z_1(t - n)\,,$$

provided c takes a particular value that you should find.

Q4, (STEP III, 2013, Q7)

(i) Let $y(x)$ be a solution of the differential equation $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + y^3 = 0$ with $y = 1$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ at $x = 0$, and let

$$\mathrm{E}(x) = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \tfrac{1}{2}y^4\,.$$

Show by differentiation that E is constant and deduce that $|y(x)| \leqslant 1$ for all x.

(ii) Let $v(x)$ be a solution of the differential equation $\dfrac{\mathrm{d}^2v}{\mathrm{d}x^2} + x\dfrac{\mathrm{d}v}{\mathrm{d}x} + \sinh v = 0$ with $v = \ln 3$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x} = 0$ at $x = 0$, and let

$$\mathrm{E}(x) = \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 + 2\cosh v\,.$$

Show that $\dfrac{\mathrm{dE}}{\mathrm{d}x} \leqslant 0$ for $x \geqslant 0$ and deduce that $\cosh v(x) \leqslant \frac{5}{3}$ for $x \geqslant 0$.

(iii) Let $w(x)$ be a solution of the differential equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}x^2} + (5\cosh x - 4\sinh x - 3)\frac{\mathrm{d}w}{\mathrm{d}x} + (w\cosh w + 2\sinh w) = 0$$

with $\dfrac{\mathrm{d}w}{\mathrm{d}x} = \dfrac{1}{\sqrt{2}}$ and $w = 0$ at $x = 0$. Show that $\cosh w(x) \leqslant \frac{5}{4}$ for $x \geqslant 0$.

Q5, (2018, Q4)

Show that the second-order differential equation

$$x^2y'' + (1-2p)x\,y' + (p^2-q^2)\,y = \mathrm{f}(x)\,,$$

where p and q are constants, can be written in the form

$$x^a\big(x^b(x^cy)'\big)' = \mathrm{f}(x)\,, \qquad (*)$$

where a, b and c are constants.

(i) Use $(*)$ to derive the general solution of the equation

$$x^2y'' + (1-2p)xy' + (p^2-q^2)y = 0$$

in the different cases that arise according to the values of p and q.

(ii) Use $(*)$ to derive the general solution of the equation

$$x^2y'' + (1-2p)xy' + p^2y = x^n$$

in the different cases that arise according to the values of p and n.