Step III – Polar Coordinates

STEP III Specification

Polar coordinates

Understand and use polar coordinates and be able to convert between polar and Cartesian coordinates.

(It will be assumed that $r \geq 0$; the range of θ will be given if appropriate.)

Sketch curves with r given as a function of θ, including the use of trigonometric functions.

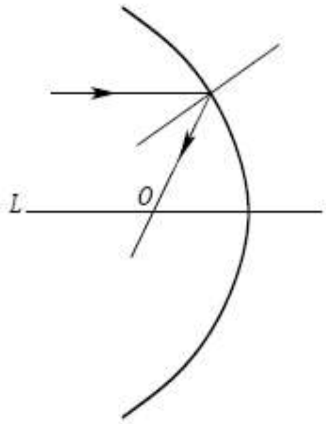
Find the area enclosed by a polar curve.

Q1, (STEP III, 2006, Q6)

Show that in polar coordinates the gradient of any curve at the point (r, θ) is

$$\frac{\dfrac{\mathrm{d}r}{\mathrm{d}\theta}\tan\theta + r}{\dfrac{\mathrm{d}r}{\mathrm{d}\theta} - r\tan\theta}.$$

A mirror is designed so that if an incident ray of light is parallel to a fixed line L the reflected ray passes through a fixed point O on L. Prove that the mirror intersects any plane containing L in a parabola. You should assume that the angle between the incident ray and the normal to the mirror is the same as the angle between the reflected ray and the normal.



Q2, (STEP III, 2017, Q5)

The point with cartesian coordinates (x, y) lies on a curve with polar equation $r = \mathrm{f}(\theta)$. Find an expression for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $\mathrm{f}(\theta)$, $\mathrm{f}'(\theta)$ and $\tan\theta$.

Two curves, with polar equations $r = \mathrm{f}(\theta)$ and $r = \mathrm{g}(\theta)$, meet at right angles. Show that where they meet

$$\mathrm{f}'(\theta)\mathrm{g}'(\theta) + \mathrm{f}(\theta)\mathrm{g}(\theta) = 0\,.$$

The curve C has polar equation $r = \mathrm{f}(\theta)$ and passes through the point given by $r = 4$, $\theta = -\frac{1}{2}\pi$. For each positive value of a, the curve with polar equation $r = a(1 + \sin\theta)$ meets C at right angles. Find $\mathrm{f}(\theta)$.

Sketch on a single diagram the three curves with polar equations $r = 1+\sin\theta$, $r = 4(1+\sin\theta)$ and $r = \mathrm{f}(\theta)$.

Q3, (STEP III, 2011, Q5)

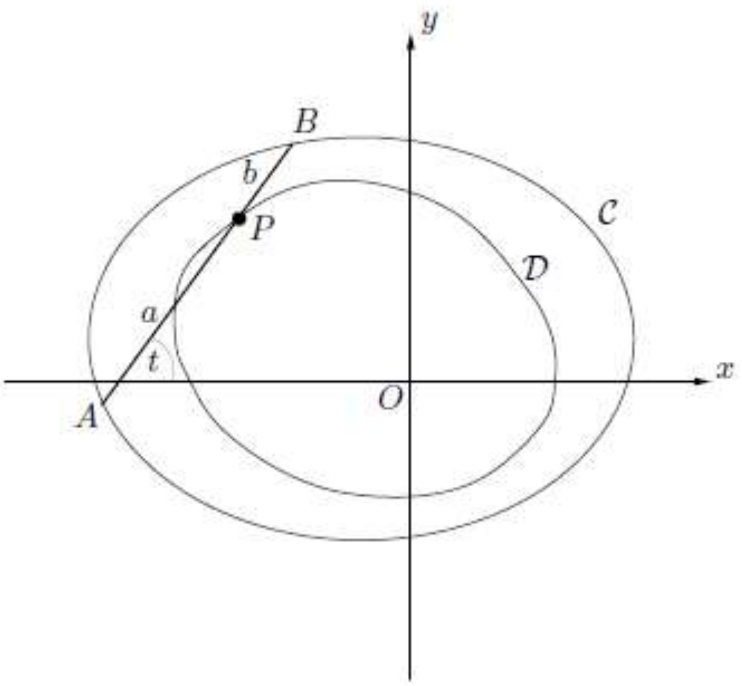
A movable point P has cartesian coordinates (x, y), where x and y are functions of t. The polar coordinates of P with respect to the origin O are r and θ. Starting with the expression

$$\tfrac{1}{2}\int r^2\,\mathrm{d}\theta$$

for the area swept out by OP, obtain the equivalent expression

$$\tfrac{1}{2}\int \left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t\,. \qquad (*)$$

The ends of a thin straight rod AB lie on a closed convex curve $\mathcal{C}$. The point P on the rod is a fixed distance a from A and a fixed distance b from B. The angle between AB and the positive x direction is t. As A and B move anticlockwise round $\mathcal{C}$, the angle t increases from 0 to 2π and P traces a closed convex curve $\mathcal{D}$ inside $\mathcal{C}$, with the origin O lying inside $\mathcal{D}$, as shown in the diagram.



Let (x, y) be the coordinates of P. Write down the coordinates of A and B in terms of a, b, x, y and t.

The areas swept out by OA, OB and OP are denoted by [A], [B] and [P], respectively. Show, using $(*)$, that

$$[A] = [P] + \pi a^2 - af$$

where

$$f = \tfrac{1}{2}\int_0^{2\pi}\left(\left(x + \frac{\mathrm{d}y}{\mathrm{d}t}\right)\cos t + \left(y - \frac{\mathrm{d}x}{\mathrm{d}t}\right)\sin t\right)\mathrm{d}t\,.$$

Obtain a corresponding expression for [B] involving b. Hence show that the area between the curves $\mathcal{C}$ and $\mathcal{D}$ is πab.

Q4, (STEP III, 2015, Q3)

In this question, r and θ are polar coordinates with $r \geqslant 0$ and $-\pi < \theta \leqslant \pi$, and a and b are positive constants.

Let L be a fixed line and let A be a fixed point not lying on L. Then the locus of points that are a fixed distance (call it d) from L measured along lines through A is called a *conchoid of Nicomedes.*

(i) Show that if

$$|r - a\sec\theta| = b\,, \qquad (*)$$

where $a > b$, then $\sec\theta > 0$. Show that all points with coordinates satisfying $(*)$ lie on a certain conchoid of Nicomedes (you should identify L, d and A). Sketch the locus of these points.

(ii) In the case $a < b$, sketch the curve (including the loop for which $\sec\theta < 0$) given by

$$|r - a\sec\theta| = b\,.$$

Find the area of the loop in the case $a = 1$ and $b = 2$.

[Note: $\int \sec\theta \,\mathrm{d}\theta = \ln|\sec\theta + \tan\theta| + C\,$.]