STEP III – Maclaurin Series

Further algebra and functions

Find the Maclaurin series of a function including the general term.

Know and use the Maclaurin series for e^x, $\ln(1+x)$, $\sin x$, $\cos x$, and $(1 + x)^n$, and be aware of the range of values of x for which they are valid (proof not required).

Q1, (STEP III, 2006, Q4)

The function f satisfies the identity

$$f(x) + f(y) \equiv f(x + y) \qquad (*)$$

for all x and y. Show that $2f(x) \equiv f(2x)$ and deduce that $f''(0) = 0$. By considering the Maclaurin series for $f(x)$, find the most general function that satisfies $(*)$.
[*Do not consider issues of existence or convergence of Maclaurin series in this question.*]

(i) By considering the function G, defined by $\ln\big(g(x)\big) = G(x)$, find the most general function that, for all x and y, satisfies the identity

$$g(x)g(y) \equiv g(x + y)\,.$$

(ii) By considering the function H, defined by $h(e^u) = H(u)$, find the most general function that satisfies, for all positive x and y, the identity

$$h(x) + h(y) \equiv h(xy)\,.$$

(iii) Find the most general function t that, for all x and y, satisfies the identity

$$t(x) + t(y) \equiv t(z)\,,$$

where $z = \dfrac{x + y}{1 - xy}$.

Q2, (STEP III, 2010, Q7)

Given that $y = \cos(m \arcsin x)$, for $|x| < 1$, prove that

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + m^2y = 0\,.$$

Obtain a similar equation relating $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$, and a similar equation relating $\frac{\mathrm{d}^4y}{\mathrm{d}x^4}$, $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

Conjecture and prove a relation between $\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}$, $\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}}$ and $\frac{\mathrm{d}^ny}{\mathrm{d}x^n}$.

Obtain the first three non-zero terms of the Maclaurin series for y. Show that, if m is an even integer, $\cos m\theta$ may be written as a polynomial in $\sin\theta$ beginning

$$1 - \frac{m^2\sin^2\theta}{2!} + \frac{m^2(m^2-2^2)\sin^4\theta}{4!} - \cdots\,. \qquad (|\theta| < \tfrac{1}{2}\pi)$$

State the degree of the polynomial.

Q3, (STEP III, 2013, Q2)

In this question, you may ignore questions of convergence.

Let $y = \frac{\arcsin x}{\sqrt{1-x^2}}$. Show that

$$(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x} - xy - 1 = 0$$

and prove that, for any positive integer n,

$$(1-x^2)\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}} - (2n+3)x\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}} - (n+1)^2\frac{\mathrm{d}^ny}{\mathrm{d}x^n} = 0\,.$$

Hence obtain the Maclaurin series for $\frac{\arcsin x}{\sqrt{1-x^2}}$, giving the general term for odd and for even powers of x.

Evaluate the infinite sum

$$1 + \frac{1}{3!} + \frac{2^2}{5!} + \frac{2^2\times3^2}{7!} + \cdots + \frac{2^2\times3^2\times\cdots\times n^2}{(2n+1)!} + \cdots\,.$$
