STEP III – Integration With Hyperbolic Functions

STEP III Specification

Hyperbolic functions

Know, understand and use the definitions of hyperbolic functions $\sinh x$, $\cosh x$, $\tanh x$, $\mathbf{sech}\,x$, $\mathbf{cosech}\,x$, $\mathbf{coth}\,x$ including their domains and ranges, and be able to sketch their graphs.

Know, understand and use standard formulae for algebraic relations between hyperbolic functions, such as $\mathbf{\cosh^2 x - \sinh^2 x = 1}$.

Differentiate and integrate hyperbolic functions.

Understand and be able to use the definitions of the inverse hyperbolic functions and their domains and ranges.

Derive and use the logarithmic forms of the inverse hyperbolic functions.

Integrate functions of the form $(x^2+1)^{-\frac{1}{2}}$ and $(x^2-1)^{-\frac{1}{2}}$ and be able to choose substitutions to integrate associated functions.

Q1, (STEP III, 2004, Q1)

Show that

$$\int_0^a \frac{\sinh x}{2\cosh^2 x - 1}\,\mathrm{d}x = \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}\cosh a - 1}{\sqrt{2}\cosh a + 1}\right) + \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

and find

$$\int_0^a \frac{\cosh x}{1 + 2\sinh^2 x}\,\mathrm{d}x\,.$$

Hence show that

$$\int_0^\infty \frac{\cosh x - \sinh x}{1 + 2\sinh^2 x}\,\mathrm{d}x = \frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\,.$$

By substituting $u = \mathrm{e}^x$ in this result, or otherwise, find

$$\int_1^\infty \frac{1}{1+u^4}\,\mathrm{d}u\,.$$

Q2, (STEP III, 2010, Q2)

In this question, a is a positive constant.

(i) Express $\cosh a$ in terms of exponentials.

By using partial fractions, prove that

$$\int_0^1 \frac{1}{x^2 + 2x\cosh a + 1}\,dx = \frac{a}{2\sinh a}.$$

(ii) Find, expressing your answers in terms of hyperbolic functions,

$$\int_1^\infty \frac{1}{x^2 + 2x\sinh a - 1}\,dx$$

and

$$\int_0^\infty \frac{1}{x^4 + 2x^2\cosh a + 1}\,dx.$$

Q3, (STEP III, 2011, Q6)

The definite integrals T, U, V and X are defined by

$$T = \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\operatorname{artanh} t}{t}\,dt, \qquad U = \int_{\ln 2}^{\ln 3} \frac{u}{2\sinh u}\,du,$$

$$V = -\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\ln v}{1 - v^2}\,dv, \qquad X = \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}\ln 3} \ln(\coth x)\,dx.$$

Show, without evaluating any of them, that T, U, V and X are all equal.

Q4, (STEP III, 2014, Q2)

(i) Show, by means of the substitution $u = \cosh x$, that

$$\int \frac{\sinh x}{\cosh 2x}\,dx = \frac{1}{2\sqrt{2}} \ln\left|\frac{\sqrt{2}\cosh x - 1}{\sqrt{2}\cosh x + 1}\right| + C.$$

(ii) Use a similar substitution to find an expression for

$$\int \frac{\cosh x}{\cosh 2x}\,dx.$$

(iii) Using parts (i) and (ii) above, show that

$$\int_0^1 \frac{1}{1+u^4}\,du = \frac{\pi + 2\ln(\sqrt{2}+1)}{4\sqrt{2}}.$$
