STEP III – Hyperbolic Functions

STEP III Specification

Hyperbolic functions

Know, understand and use the definitions of hyperbolic functions $\sinh x$, $\cosh x$, $\tanh x$, $\mathbf{sech}\,\boldsymbol{x}$, $\mathbf{cosech}\,\boldsymbol{x}$, $\mathbf{coth}\,\boldsymbol{x}$ including their domains and ranges, and be able to sketch their graphs.

Know, understand and use standard formulae for algebraic relations between hyperbolic functions, such as $\mathbf{\cosh^2 x - \sinh^2 x = 1}$.

Differentiate and integrate hyperbolic functions.

Understand and be able to use the definitions of the inverse hyperbolic functions and their domains and ranges.

Derive and use the logarithmic forms of the inverse hyperbolic functions.

Integrate functions of the form $(x^2+1)^{-\frac{1}{2}}$ and $(x^2-1)^{-\frac{1}{2}}$ and be able to choose substitutions to integrate associated functions.

Q1, (STEP III, 2005, Q6)

In this question, you may use without proof the results

$$4\cosh^3 y - 3\cosh y = \cosh(3y) \quad \text{and} \quad \operatorname{arcosh} y = \ln\left(y + \sqrt{y^2-1}\right).$$

[**Note:** arcosh$y$ is another notation for $\cosh^{-1} y$]

Show that the equation $x^3 - 3a^2x = 2a^3\cosh T$ is satisfied by $2a\cosh\left(\frac{1}{3}T\right)$ and hence that, if $c^2 \geqslant b^3$, one of the roots of the equation $x^3 - 3bx = 2c$ is $u + \dfrac{b}{u}$, where $u = \left(c + \sqrt{c^2-b^3}\right)^{\frac{1}{3}}$.

Show that the other two roots of the equation $x^3 - 3bx = 2c$ are the roots of the quadratic equation $x^2 + (u + \dfrac{b}{u})x + u^2 + \dfrac{b^2}{u^2} - b = 0$, and find these roots in terms of u, b and ω, where $\omega = \frac{1}{2}(-1 + i\sqrt{3})$.

Solve completely the equation $x^3 - 6x = 6$.

Q2, (STEP III, 2008, Q4)

(i) Show, with the aid of a sketch, that $y > \tanh(y/2)$ for $y > 0$ and deduce that

$$\operatorname{arcosh} x > \frac{x-1}{\sqrt{x^2-1}} \quad \text{for} \quad x > 1. \qquad (*)$$

(ii) By integrating $(*)$, show that $\operatorname{arcosh} x > 2\dfrac{x-1}{\sqrt{x^2-1}}$ for $x > 1$.

(iii) Show that $\operatorname{arcosh} x > 3\dfrac{\sqrt{x^2-1}}{x+2}$ for $x > 1$.

[**Note:** $\operatorname{arcosh} x$ is another notation for $\cosh^{-1} x$.]

Q3, (STEP III, 2014, Q6)

Starting from the result that

$$h(t) > 0 \text{ for } 0 < t < x \Longrightarrow \int_0^x h(t)\,dt > 0\,,$$

show that, if $f''(t) > 0$ for $0 < t < x_0$ and $f(0) = f'(0) = 0$, then $f(t) > 0$ for $0 < t < x_0$.

(i) Show that, for $0 < x < \frac{1}{2}\pi$,

$$\cos x \cosh x < 1\,.$$

(ii) Show that, for $0 < x < \frac{1}{2}\pi$,

$$\frac{1}{\cosh x} < \frac{\sin x}{x} < \frac{x}{\sinh x}\,.$$

Q4, (STEP III, 2016, Q6)

Show, by finding R and γ, that $A\sinh x + B\cosh x$ can be written in the form $R\cosh(x+\gamma)$ if $B > A > 0$. Determine the corresponding forms in the other cases that arise, for $A > 0$, according to the value of B.

Two curves have equations $y = \operatorname{sech} x$ and $y = a\tanh x + b$, where $a > 0$.

(i) In the case $b > a$, show that if the curves intersect then the x-coordinates of the points of intersection can be written in the form

$$\pm\,\text{arcosh}\left(\frac{1}{\sqrt{b^2 - a^2}}\right) - \text{artanh}\,\frac{a}{b}.$$

(ii) Find the corresponding result in the case $a > b > 0$.

(iii) Find necessary and sufficient conditions on a and b for the curves to intersect at two distinct points.

(iv) Find necessary and sufficient conditions on a and b for the curves to touch and, given that they touch, express the y-coordinate of the point of contact in terms of a.
