STEP III – First Order Differential Equations

STEP III Specification

Differential equations

Find and use an integrating factor to solve differential equations of the form $\frac{dy}{dx} + P(x)y = Q(x)$; recognise when it is appropriate to do so.

Find both general and particular solutions to differential equations, ***including by methods that will be indicated if appropriate.***

Use differential equations in modelling in kinematics and in other contexts.

Use given substitutions to transform differential equations.

Q1, (STEP III, 2004, Q8)

Show that if

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)y + \frac{\mathrm{g}(x)}{y}$$

then the substitution $u = y^2$ gives a linear differential equation for $u(x)$.

Hence or otherwise solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} - \frac{1}{y}.$$

Determine the solution curves of this equation which pass through $(1\,,1)$, $(2\,,2)$ and $(4\,,4)$ and sketch graphs of all three curves on the same axes.

Q2, (STEP III, 2006, Q7)

(i) Solve the equation $u^2 + 2u\sinh x - 1 = 0$ giving u in terms of x.

Find the solution of the differential equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x}\sinh x - 1 = 0$$

that satisfies $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ at $x = 0$.

(ii) Find the solution, not identically zero, of the differential equation

$$\sinh y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x} - \sinh y = 0$$

that satisfies $y = 0$ at $x = 0$, expressing your solution in the form $\cosh y = \mathrm{f}(x)$. Show that the asymptotes to the solution curve are $y = \pm(-x + \ln 4)$.

Q3, (STEP III, 2008, Q6)

In this question, p denotes $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(i) Given that

$$y = p^2 + 2xp\,,$$

show by differentiating with respect to x that

$$\frac{\mathrm{d}x}{\mathrm{d}p} = -2 - \frac{2x}{p}.$$

Hence show that $x = -\frac{2}{3}p + Ap^{-2}$, where A is an arbitrary constant.

Find y in terms of x if $p = -3$ when $x = 2$.

(ii) Given instead that

$$y = 2xp + p\ln p\,,$$

and that $p = 1$ when $x = -\frac{1}{4}$, show that $x = -\frac{1}{2}\ln p - \frac{1}{4}$ and find y in terms of x.