STEP III – Complex Numbers

STEP III Specification

Further Complex numbers

Know and understand de Moivre's theorem and use it to find multiple angle formulae and sums of series.

Know and use the definition $e^{i\theta} = \cos\theta + i\sin\theta$ and the form $z = re^{i\theta}$

Find the n distinct nth roots of $re^{i\theta}$ for $r \neq 0$ and know that they form the vertices of a regular n-gon in the Argand diagram.

Use complex numbers, including complex roots of unity, to solve geometric problems.

Q1, (STEP III, 2006, Q5)

Show that the distinct complex numbers α, β and γ represent the vertices of an equilateral triangle (in clockwise or anti-clockwise order) if and only if

$$\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta = 0\,.$$

Show that the roots of the equation

$$z^3 + az^2 + bz + c = 0 \qquad (*)$$

represent the vertices of an equilateral triangle if and only if $a^2 = 3b$.

Under the transformation $z = pw + q$, where p and q are given complex numbers with $p \neq 0$, the equation $(*)$ becomes

$$w^3 + Aw^2 + Bw + C = 0\,. \qquad (**)$$

Show that if the roots of equation $(*)$ represent the vertices of an equilateral triangle, then the roots of equation $(**)$ also represent the vertices of an equilateral triangle.

Q2, (STEP III, 2011, Q3)

Show that, provided $q^2 \neq 4p^3$, the polynomial

$$x^3 - 3px + q \qquad (p \neq 0,\ q \neq 0)$$

can be written in the form

$$a(x-\alpha)^3 + b(x-\beta)^3\,,$$

where α and β are the roots of the quadratic equation $pt^2 - qt + p^2 = 0$, and a and b are constants which you should express in terms of α and β.

Hence show that one solution of the equation $x^3 - 24x + 48 = 0$ is

$$x = \frac{2(2 - 2^{\frac{1}{3}})}{1 - 2^{\frac{1}{3}}}$$

and obtain similar expressions for the other two solutions in terms of ω, where $\omega = e^{2\pi i/3}$.

Find also the roots of $x^3 - 3px + q = 0$ when $p = r^2$ and $q = 2r^3$ for some non-zero constant r.

Q3, (STEP III, 2009, Q6)

Show that $\left|e^{i\beta} - e^{i\alpha}\right| = 2\sin\frac{1}{2}(\beta - \alpha)$ for $0 < \alpha < \beta < 2\pi$. Hence show that

$$\left|e^{i\alpha} - e^{i\beta}\right| \left|e^{i\gamma} - e^{i\delta}\right| + \left|e^{i\beta} - e^{i\gamma}\right| \left|e^{i\alpha} - e^{i\delta}\right| = \left|e^{i\alpha} - e^{i\gamma}\right| \left|e^{i\beta} - e^{i\delta}\right|,$$

where $0 < \alpha < \beta < \gamma < \delta < 2\pi$.

Interpret this result as a theorem about cyclic quadrilaterals.

Q4, (STEP II, 2013, Q4)

Show that $(z - e^{i\theta})(z - e^{-i\theta}) = z^2 - 2z\cos\theta + 1$.

Write down the $(2n)$th roots of -1 in the form $e^{i\theta}$, where $-\pi < \theta \leqslant \pi$, and deduce that

$$z^{2n} + 1 = \prod_{k=1}^{n} \left(z^2 - 2z\cos\left(\tfrac{(2k-1)\pi}{2n}\right) + 1\right).$$

Here, n is a positive integer, and the $\prod$ notation denotes the product.

(i) By substituting $z = i$ show that, when n is even,

$$\cos\left(\tfrac{\pi}{2n}\right)\cos\left(\tfrac{3\pi}{2n}\right)\cos\left(\tfrac{5\pi}{2n}\right)\cdots\cos\left(\tfrac{(2n-1)\pi}{2n}\right) = (-1)^{\frac{1}{2}n}2^{1-n}.$$

(ii) Show that, when n is odd,

$$\cos^2\left(\tfrac{\pi}{2n}\right)\cos^2\left(\tfrac{3\pi}{2n}\right)\cos^2\left(\tfrac{5\pi}{2n}\right)\cdots\cos^2\left(\tfrac{(n-2)\pi}{2n}\right) = n2^{1-n}.$$

You may use without proof the fact that $1 + z^{2n} = (1 + z^2)(1 - z^2 + z^4 - \cdots + z^{2n-2})$ when n is odd.

Q5 (STEP III, 2016, Q7)

Let $\omega = e^{2\pi i/n}$, where n is a positive integer. Show that, for any complex number z,

$$(z-1)(z-\omega)\cdots(z-\omega^{n-1}) = z^n - 1.$$

The points $X_0, X_1, \ldots, X_{n-1}$ lie on a circle with centre O and radius 1, and are the vertices of a regular polygon.

(i) The point P is equidistant from X_0 and X_1. Show that, if n is even,

$$|PX_0| \times |PX_1| \times \cdots \times |PX_{n-1}| = |OP|^n + 1,$$

where $|PX_k|$ denotes the distance from P to X_k.

Give the corresponding result when n is odd. (There are two cases to consider.)

(ii) Show that

$$|X_0X_1| \times |X_0X_2| \times \cdots \times |X_0X_{n-1}| = n.$$

Q6, (STEP III, 2017, Q2)

The transformation R in the complex plane is a rotation (anticlockwise) by an angle θ about the point represented by the complex number a. The transformation S in the complex plane is a rotation (anticlockwise) by an angle ϕ about the point represented by the complex number b.

(i) The point P is represented by the complex number z. Show that the image of P under R is represented by

$$e^{i\theta}z + a(1 - e^{i\theta}).$$

(ii) Show that the transformation SR (equivalent to R followed by S) is a rotation about the point represented by c, where

$$c \sin \tfrac{1}{2}(\theta + \phi) = a\, e^{i\phi/2} \sin \tfrac{1}{2}\theta + b\, e^{-i\theta/2} \sin \tfrac{1}{2}\phi,$$

provided $\theta + \phi \neq 2n\pi$ for any integer n.

What is the transformation SR if $\theta + \phi = 2\pi$?

(iii) Under what circumstances is $RS = SR$?
