<u>**STEP II – Vectors**</u>

<u>**STEP II Specfication**</u>

**Further vectors**

***Understand and use the vector and Cartesian forms of an equation of a straight line in 3-D.***

***Understand and use the scalar product of two vectors, including geometrical interpretation and formal algebraic manipulation; for example,*** $\mathbf{a}.(\mathbf{b}+\mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$

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<u>**Q1, (STEP I, 2010, Q7)**</u>

Relative to a fixed origin $O$, the points $A$ and $B$ have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. (The points $O$, $A$ and $B$ are not collinear.) The point $C$ has position vector $\mathbf{c}$ given by

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}\,,$$

where $\alpha$ and $\beta$ are positive constants with $\alpha + \beta < 1$. The lines $OA$ and $BC$ meet at the point $P$ with position vector $\mathbf{p}$ and the lines $OB$ and $AC$ meet at the point $Q$ with position vector $\mathbf{q}$. Show that

$$\mathbf{p} = \frac{\alpha\mathbf{a}}{1-\beta}\,,$$

and write down $\mathbf{q}$ in terms of $\alpha$, $\beta$ and $\mathbf{b}$.

Show further that the point $R$ with position vector $\mathbf{r}$ given by

$$\mathbf{r} = \frac{\alpha\mathbf{a} + \beta\mathbf{b}}{\alpha+\beta}\,,$$

lies on the lines $OC$ and $AB$.

The lines $OB$ and $PR$ intersect at the point $S$. Prove that $\dfrac{OQ}{BQ} = \dfrac{OS}{BS}$.

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<u>**Q2, (STEP I, 2013, Q3)**</u>

For any two points $X$ and $Y$, with position vectors $\mathbf{x}$ and $\mathbf{y}$ respectively, $X * Y$ is defined to be the point with position vector $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$, where $\lambda$ is a fixed number.

**(i)** If $X$ and $Y$ are distinct, show that $X*Y$ and $Y*X$ are distinct unless $\lambda$ takes a certain value (which you should state).

**(ii)** Under what conditions are $(X*Y)*Z$ and $X*(Y*Z)$ distinct?

**(iii)** Show that, for any points $X$, $Y$ and $Z$,

$$(X*Y)*Z = (X*Z)*(Y*Z)$$

and obtain the corresponding result for $X*(Y*Z)$.

**(iv)** The points $P_1$, $P_2$, ... are defined by $P_1 = X*Y$ and, for $n \geqslant 2$, $P_n = P_{n-1}*Y$. Given that $X$ and $Y$ are distinct and that $0<\lambda<1$, find the ratio in which $P_n$ divides the line segment $XY$.

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**Q3, (STEP I, 2014, Q7)**

In the triangle $OAB$, the point $D$ divides the side $BO$ in the ratio $r : 1$ (so that $BD = rDO$), and the point $E$ divides the side $OA$ in the ratio $s : 1$ (so that $OE = sEA$), where $r$ and $s$ are both positive.

**(i)** The lines $AD$ and $BE$ intersect at $G$. Show that

$$\mathbf{g} = \frac{rs}{1+r+rs}\,\mathbf{a} + \frac{1}{1+r+rs}\,\mathbf{b}\,,$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{g}$ are the position vectors with respect to $O$ of $A$, $B$ and $G$, respectively.

**(ii)** The line through $G$ and $O$ meets $AB$ at $F$. Given that $F$ divides $AB$ in the ratio $t : 1$, find an expression for $t$ in terms of $r$ and $s$.

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**Q4, (STEP I, 2015, Q6)**

The vertices of a plane quadrilateral are labelled $A$, $B$, $A'$ and $B'$, in clockwise order. A point $O$ lies in the same plane and within the quadrilateral. The angles $AOB$ and $A'OB'$ are right angles, and $OA = OB$ and $OA' = OB'$.

Use position vectors relative to $O$ to show that the midpoints of $AB$, $BA'$, $A'B'$ and $B'A$ are the vertices of a square.

Given that the lengths of $OA$ and $OA'$ are fixed (and the conditions of the first paragraph still hold), find the value of angle $BOA'$ for which the area of the square is greatest.

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**Q5, (STEP I, 2016, Q6)**

The sides $OA$ and $CB$ of the quadrilateral $OABC$ are parallel. The point $X$ lies on $OA$, between $O$ and $A$. The position vectors of $A$, $B$, $C$ and $X$ relative to the origin $O$ are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{x}$, respectively. Explain why $\mathbf{c}$ and $\mathbf{x}$ can be written in the form

$$\mathbf{c} = k\mathbf{a} + \mathbf{b} \qquad \text{and} \qquad \mathbf{x} = m\mathbf{a}\,,$$

where $k$ and $m$ are scalars, and state the range of values that each of $k$ and $m$ can take.

The lines $OB$ and $AC$ intersect at $D$, the lines $XD$ and $BC$ intersect at $Y$ and the lines $OY$ and $AB$ intersect at $Z$. Show that the position vector of $Z$ relative to $O$ can be written as

$$\frac{\mathbf{b} + mk\mathbf{a}}{mk+1}\,.$$

The lines $DZ$ and $OA$ intersect at $T$. Show that

$$OT \times OA = OX \times TA \qquad \text{and} \qquad \frac{1}{OT} = \frac{1}{OX} + \frac{1}{OA}\,,$$

where, for example, $OT$ denotes the length of the line joining $O$ and $T$.

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**Q6, (STEP II, 2018, Q7)**

The points $O$, $A$ and $B$ are the vertices of an acute-angled triangle. The points $M$ and $N$ lie on the sides $OA$ and $OB$ respectively, and the lines $AN$ and $BM$ intersect at $Q$. The position vector of $A$ with respect to $O$ is $\mathbf{a}$, and the position vectors of the other points are labelled similarly.

Given that $|MQ| = \mu|QB|$, and that $|NQ| = \nu|QA|$, where $\mu$ and $\nu$ are positive and $\mu\nu < 1$, show that

$$\mathbf{m} = \frac{(1+\mu)\nu}{1+\nu}\,\mathbf{a}\,.$$

The point $L$ lies on the side $OB$, and $|OL| = \lambda|OB|$. Given that $ML$ is parallel to $AN$, express $\lambda$ in terms of $\mu$ and $\nu$.

What is the geometrical significance of the condition $\mu\nu < 1$?

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