Sequences and Series Expansion

STEP II Specification

Further algebra and functions

Know and use partial fractions in which the denominator may include quadratic factors of the form $ax^2 + c$ for $c > 0$, and in which the degree of the numerator may be equal to, or exceed, the degree of the denominator.

Understand and use the method of differences for summation of series, including the use of partial fractions.

Recognise and use the series expansion of e^x.

Q1, (STEP II, 2008, Q2)

Let a_n be the coefficient of x^n in the series expansion, in ascending powers of x, of

$$\frac{1+x}{(1-x)^2(1+x^2)},$$

where $|x| < 1$. Show, using partial fractions, that either $a_n = n+1$ or $a_n = n+2$ according to the value of n.

Hence find a decimal approximation, to nine significant figures, for the fraction $\frac{11\,000}{8181}$.
[You are not required to justify the accuracy of your approximation.]

Q2, (STEP II, 2009, Q6)

The Fibonacci sequence $F_1, F_2, F_3, \ldots$ is defined by $F_1 = 1$, $F_2 = 1$ and

$$F_{n+1} = F_n + F_{n-1} \qquad (n \geqslant 2).$$

Write down the values of $F_3, F_4, \ldots, F_{10}$.

Let $S = \displaystyle\sum_{i=1}^{\infty} \frac{1}{F_i}$.

(i) Show that $\dfrac{1}{F_i} > \dfrac{1}{2F_{i-1}}$ for $i \geqslant 4$ and deduce that $S > 3$.

Show also that $S < 3\frac{2}{3}$.

(ii) Show further that $3.2 < S < 3.5$.

Q3, (STEP II, 2010, Q3)

The first four terms of a sequence are given by $F_0 = 0$, $F_1 = 1$, $F_2 = 1$ and $F_3 = 2$. The general term is given by

$$F_n = a\lambda^n + b\mu^n \,, \qquad (*)$$

where a, b, λ and μ are independent of n, and a is positive.

(i) Show that $\lambda^2 + \lambda\mu + \mu^2 = 2$, and find the values of λ, μ, a and b.

(ii) Use $(*)$ to evaluate F_6.

(iii) Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{F_n}{2^{n+1}}$.

Q4, (STEP II, 2011, Q7)

The two sequences a_0, a_1, a_2, ... and b_0, b_1, b_2, ... have general terms

$$a_n = \lambda^n + \mu^n \quad \text{and} \quad b_n = \lambda^n - \mu^n \,,$$

respectively, where $\lambda = 1 + \sqrt{2}$ and $\mu = 1 - \sqrt{2}$.

(i) Show that $\displaystyle\sum_{r=0}^{n} b_r = -\sqrt{2} + \frac{1}{\sqrt{2}}\, a_{n+1}$, and give a corresponding result for $\displaystyle\sum_{r=0}^{n} a_r$.

(ii) Show that, if n is odd,

$$\sum_{m=0}^{2n} \left(\sum_{r=0}^{m} a_r \right) = \tfrac{1}{2} b_{n+1}^2 \,,$$

and give a corresponding result when n is even.

(iii) Show that, if n is even,

$$\left(\sum_{r=0}^{n} a_r \right)^2 - \sum_{r=0}^{n} a_{2r+1} = 2 \,,$$

and give a corresponding result when n is odd.

Q5, (STEP II, 2015, Q1)

(i) By use of calculus, show that $x - \ln(1+x)$ is positive for all positive x. Use this result to show that

$$\sum_{k=1}^{n} \frac{1}{k} > \ln(n+1) \,.$$

(ii) By considering $x + \ln(1-x)$, show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < 1 + \ln 2 \,.$$

Q6, (STEP II 2012, Q4)

In this question, you may assume that the infinite series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots$$

is valid for $|x| < 1$.

(i) Let n be an integer greater than 1. Show that, for any positive integer k,

$$\frac{1}{(k+1)n^{k+1}} < \frac{1}{kn^k}.$$

Hence show that $\ln\left(1+\frac{1}{n}\right) < \frac{1}{n}$. Deduce that

$$\left(1+\frac{1}{n}\right)^n < \mathrm{e}.$$

(ii) Show, using an expansion in powers of $\frac{1}{y}$, that $\ln\left(\frac{2y+1}{2y-1}\right) > \frac{1}{y}$ for $y > \frac{1}{2}$.

Deduce that, for any positive integer n,

$$\mathrm{e} < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}.$$

(iii) Use parts (i) and (ii) to show that as $n \to \infty$

$$\left(1+\frac{1}{n}\right)^n \to \mathrm{e}.$$

Q7, (STEP II, 2013, Q6)

In this question, the following theorem may be used.

Let u_1, u_2, ... be a sequence of (real) numbers. If the sequence is bounded above (that is, $u_n \leqslant b$ for all n, where b is some fixed number) and increasing (that is, $u_n \geqslant u_{n-1}$ for all n), then the sequence tends to a limit (that is, converges).

The sequence u_1, u_2, ... is defined by $u_1 = 1$ and

$$u_{n+1} = 1 + \frac{1}{u_n} \qquad (n \geqslant 1). \qquad (*)$$

(i) Show that, for $n \geqslant 3$,

$$u_{n+2} - u_n = \frac{u_n - u_{n-2}}{(1+u_n)(1+u_{n-2})}.$$

(ii) Prove, by induction or otherwise, that $1 \leqslant u_n \leqslant 2$ for all n.

(iii) Show that the sequence u_1, u_3, u_5, ... tends to a limit, and that the sequence u_2, u_4, u_6, ... tends to a limit. Find these limits and deduce that the sequence u_1, u_2, u_3, ... tends to a limit.

Would this conclusion change if the sequence were defined by $(*)$ and $u_1 = 3$?

Q8, (STEP II, 2017, Q6)

Let

$$S_n = \sum_{r=1}^{n} \frac{1}{\sqrt{r}} \,,$$

where n is a positive integer.

(i) Prove by induction that

$$S_n \leqslant 2\sqrt{n} - 1 \,.$$

(ii) Show that $(4k+1)\sqrt{k+1} > (4k+3)\sqrt{k}$ for $k \geqslant 0$.

Determine the smallest number C such that

$$S_n \geqslant 2\sqrt{n} + \frac{1}{2\sqrt{n}} - C \,.$$
