STEP II - Roots of Polynomials

Step II Specification

Further algebra and functions

Understand and use the relationship between roots and coefficients of polynomial equations up to quartic ***and higher degree*** equations.

Form a polynomial equation whose roots are a linear transformation of the roots of a given polynomial equation (of at least cubic degree).

Q1, (STEP II, 2009, Q4)

The polynomial $\mathrm{p}(x)$ is of degree 9 and $\mathrm{p}(x) - 1$ is exactly divisible by $(x-1)^5$.

(i) Find the value of $\mathrm{p}(1)$.

(ii) Show that $\mathrm{p}'(x)$ is exactly divisible by $(x-1)^4$.

(iii) Given also that $\mathrm{p}(x) + 1$ is exactly divisible by $(x+1)^5$, find $\mathrm{p}(x)$.

Q2, (STEP II, 2010, Q7)

(i) By considering the positions of its turning points, show that the curve with equation

$$y = x^3 - 3qx - q(1+q),$$

where $q > 0$ and $q \neq 1$, crosses the x-axis once only.

(ii) Given that x satisfies the cubic equation

$$x^3 - 3qx - q(1+q) = 0,$$

and that

$$x = u + q/u,$$

obtain a quadratic equation satisfied by u^3. Hence find the real root of the cubic equation in the case $q > 0$, $q \neq 1$.

(iii) The quadratic equation

$$t^2 - pt + q = 0$$

has roots α and β. Show that

$$\alpha^3 + \beta^3 = p^3 - 3qp.$$

It is given that one of these roots is the square of the other. By considering the expression $(\alpha^2 - \beta)(\beta^2 - \alpha)$, find a relationship between p and q. Given further that $q > 0$, $q \neq 1$ and p is real, determine the value of p in terms of q.

Q3, (STEP III, 2011, Q2)

The polynomial $f(x)$ is defined by

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 ,$$

where $n \geqslant 2$ and the coefficients $a_0, \ldots, a_{n-1}$ are integers, with $a_0 \neq 0$. Suppose that the equation $f(x) = 0$ has a rational root p/q, where p and q are integers with no common factor greater than 1, and $q > 0$. By considering $q^{n-1}f(p/q)$, find the value of q and deduce that any rational root of the equation $f(x) = 0$ must be an integer.

(i) Show that the nth root of 2 is irrational for $n \geqslant 2$.

(ii) Show that the cubic equation

$$x^3 - x + 1 = 0$$

has no rational roots.

(iii) Show that the polynomial equation

$$x^n - 5x + 7 = 0$$

has no rational roots for $n \geqslant 2$.

Q4, (STEP II, 2012, Q2)

If $p(x)$ and $q(x)$ are polynomials of degree m and n, respectively, what is the degree of $p(q(x))$?

(i) The polynomial $p(x)$ satisfies

$$p(p(p(x))) - 3p(x) = -2x$$

for all x. Explain carefully why $p(x)$ must be of degree 1, and find all polynomials that satisfy this equation.

(ii) Find all polynomials that satisfy

$$2p(p(x)) + 3[p(x)]^2 - 4p(x) = x^4$$

for all x.

Q5, (STEP II, 2013, Q3)

(i) Given that the cubic equation $x^3 + 3ax^2 + 3bx + c = 0$ has three distinct real roots and $c < 0$, show with the help of sketches that either exactly one of the roots is positive or all three of the roots are positive.

(ii) Given that the equation $x^3 + 3ax^2 + 3bx + c = 0$ has three distinct real positive roots show that

$$a^2 > b > 0, \quad a < 0, \quad c < 0. \qquad (*)$$

[**Hint**: Consider the turning points.]

(iii) Given that the equation $x^3 + 3ax^2 + 3bx + c = 0$ has three distinct real roots and that

$$ab < 0, \quad c > 0,$$

determine, with the help of sketches, the signs of the roots.

(iv) Show by means of an explicit example (giving values for a, b and c) that it is possible for the conditions $(*)$ to be satisfied even though the corresponding cubic equation has only one real root.

Q6, (STEP II, Q2)

Use the factor theorem to show that $a + b - c$ is a factor of

$$(a + b + c)^3 - 6(a + b + c)(a^2 + b^2 + c^2) + 8(a^3 + b^3 + c^3). \qquad (*)$$

Hence factorise $(*)$ completely.

(i) Use the result above to solve the equation

$$(x + 1)^3 - 3(x + 1)(2x^2 + 5) + 2(4x^3 + 13) = 0.$$

(ii) By setting $d + e = c$, or otherwise, show that $(a + b - d - e)$ is a factor of

$$(a + b + d + e)^3 - 6(a + b + d + e)(a^2 + b^2 + d^2 + e^2) + 8(a^3 + b^3 + d^3 + e^3)$$

and factorise this expression completely.

Hence solve the equation

$$(x + 6)^3 - 6(x + 6)(x^2 + 14) + 8(x^3 + 36) = 0.$$

Q7, (STEP III, 2017, Q3)

Let α, β, γ and δ be the roots of the quartic equation

$$x^4 + px^3 + qx^2 + rx + s = 0\,.$$

You are given that, for any such equation, $\alpha\beta + \gamma\delta$, $\alpha\gamma + \beta\delta$ and $\alpha\delta + \beta\gamma$ satisfy a cubic equation of the form

$$y^3 + Ay^2 + (pr - 4s)y + (4qs - p^2s - r^2) = 0\,.$$

Determine A.

Now consider the quartic equation given by $p = 0$, $q = 3$, $r = -6$ and $s = 10$.

(i) Find the value of $\alpha\beta + \gamma\delta$, given that it is the largest root of the corresponding cubic equation.

(ii) Hence, using the values of q and s, find the value of $(\alpha+\beta)(\gamma+\delta)$ and the value of $\alpha\beta$ given that $\alpha\beta > \gamma\delta$.

(iii) Using these results, and the values of p and r, solve the quartic equation.

Q8, (STEP II, 2018, Q1)

Show that, if k is a root of the quartic equation

$$x^4 + ax^3 + bx^2 + ax + 1 = 0\,, \qquad (*)$$

then k^{-1} is a root.

You are now given that a and b in $(*)$ are both real and are such that the roots are all real.

(i) Write down all the values of a and b for which $(*)$ has only one distinct root.

(ii) Given that $(*)$ has exactly three distinct roots, show that either $b = 2a - 2$ or $b = -2a - 2$.

(iii) Solve $(*)$ in the case $b = 2a - 2$, giving your solutions in terms of a.

Given that a and b are both real and that the roots of $(*)$ are all real, find necessary and sufficient conditions, in terms of a and b, for $(*)$ to have exactly three distinct real roots.
