STEP II - Reduction Formulae

Further calculus

Integrate using reduction formulae.

Q1, (STEP III, 2009, Q8)

Let m be a positive integer and let n be a non-negative integer.

(i) Use the result $\lim_{t\to\infty} e^{-mt}t^n = 0$ to show that

$$\lim_{x\to 0} x^m(\ln x)^n = 0\,.$$

By writing x^x as $e^{x\ln x}$ show that

$$\lim_{x\to 0} x^x = 1\,.$$

(ii) Let $I_n = \int_0^1 x^m(\ln x)^n\,dx$. Show that

$$I_{n+1} = -\frac{n+1}{m+1}I_n$$

and hence evaluate I_n.

(iii) Show that

$$\int_0^1 x^x\,dx = 1 - \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{3}\right)^3 - \left(\tfrac{1}{4}\right)^4 + \cdots\,.$$

Q2, (STEP III, 2013, Q1)

Given that $t = \tan\frac{1}{2}x$, show that $\frac{dt}{dx} = \frac{1}{2}(1+t^2)$ and $\sin x = \frac{2t}{1+t^2}$.

Hence show that

$$\int_0^{\frac{1}{2}\pi} \frac{1}{1+a\sin x}\,dx = \frac{2}{\sqrt{1-a^2}}\arctan\frac{\sqrt{1-a}}{\sqrt{1+a}} \qquad (0 < a < 1).$$

Let

$$I_n = \int_0^{\frac{1}{2}\pi} \frac{\sin^n x}{2+\sin x}\,dx \qquad (n \geqslant 0).$$

By considering $I_{n+1} + 2I_n$, or otherwise, evaluate I_3.

Q3, (STEP III, 2015, Q1)

(i) Let

$$I_n = \int_0^\infty \frac{1}{(1+u^2)^n}\,\mathrm{d}u\,,$$

where n is a positive integer. Show that

$$I_n - I_{n+1} = \frac{1}{2n} I_n$$

and deduce that

$$I_{n+1} = \frac{(2n)!\,\pi}{2^{2n+1}(n!)^2}\,.$$

(ii) Let

$$J = \int_0^\infty \mathrm{f}\big((x - x^{-1})^2\big)\,\mathrm{d}x\,,$$

where f is any function for which the integral exists. Show that

$$J = \int_0^\infty x^{-2}\mathrm{f}\big((x-x^{-1})^2\big)\,\mathrm{d}x = \frac{1}{2}\int_0^\infty (1+x^{-2})\mathrm{f}\big((x-x^{-1})^2\big)\,\mathrm{d}x = \int_0^\infty \mathrm{f}\big(u^2\big)\,\mathrm{d}u\,.$$

(iii) Hence evaluate

$$\int_0^\infty \frac{x^{2n-2}}{(x^4 - x^2 + 1)^n}\,\mathrm{d}x\,,$$

where n is a positive integer.

Q4, (STEP III, 2016, Q1)

Let

$$I_n = \int_{-\infty}^\infty \frac{1}{(x^2 + 2ax + b)^n}\,\mathrm{d}x\,,$$

where a and b are constants with $b > a^2$, and n is a positive integer.

(i) By using the substitution $x + a = \sqrt{b - a^2}\,\tan u$, or otherwise, show that

$$I_1 = \frac{\pi}{\sqrt{b-a^2}}\,.$$

(ii) Show that $2n(b-a^2)\,I_{n+1} = (2n-1)\,I_n\,.$

(iii) Hence prove by induction that

$$I_n = \frac{\pi}{2^{2n-2}(b-a^2)^{n-\frac{1}{2}}}\binom{2n-2}{n-1}\,.$$

Q5, (STEP III, 2004, Q7)

For $n = 1, 2, 3, \ldots$, let

$$I_n = \int_0^1 \frac{t^{n-1}}{(t+1)^n}\,\mathrm{d}t\,.$$

By considering the greatest value taken by $\dfrac{t}{t+1}$ for $0 \leqslant t \leqslant 1$ show that $I_{n+1} < \frac{1}{2}I_n$.

Show also that $I_{n+1} = -\dfrac{1}{n\,2^n} + I_n$.

Deduce that $I_n < \dfrac{1}{n\,2^{n-1}}$.

Prove that

$$\ln 2 = \sum_{r=1}^{n} \frac{1}{r\,2^r} + I_{n+1}$$

and hence show that $\frac{2}{3} < \ln 2 < \frac{17}{24}$.