STEP II – Poisson Distribution

Probability distributions

Know, understand and use the Poisson distribution; find probabilities using the Poisson distribution.

Calculate the mean and variance of the Poisson distribution.

Q1, (STEP II, 2005, Q13)

The number of printing errors on any page of a large book of N pages is modelled by a Poisson variate with parameter λ and is statistically independent of the number of printing errors on any other page. The number of pages in a random sample of n pages (where n is much smaller than N and $n \geqslant 2$) which contains fewer than two errors is denoted by Y. Show that $\mathrm{P}(Y = k) = \binom{n}{k} p^k q^{n-k}$ where $p = (1 + \lambda)e^{-\lambda}$ and $q = 1 - p$.

Show also that, if λ is sufficiently small,

(i) $q \approx \frac{1}{2}\lambda^2$;

(ii) the largest value of n for which $\mathrm{P}(Y = n) \geqslant 1 - \lambda$ is approximately $2/\lambda$;

(iii) $\mathrm{P}(Y > 1 \mid Y > 0) \approx 1 - n(\lambda^2/2)^{n-1}$.

Q2, (STEP II, 2007, Q14)

The discrete random variable X has a Poisson distribution with mean λ.

(i) Sketch the graph $y = (x + 1)\,\mathrm{e}^{-x}$, stating the coordinates of the turning point and the points of intersection with the axes.

It is known that $\mathrm{P}(X \geqslant 2) = 1 - p$, where p is a given number in the range $0 < p < 1$. Show that this information determines a unique value (which you should not attempt to find) of λ.

(ii) It is known (instead) that $\mathrm{P}(X = 1) = q$, where q is a given number in the range $0 < q < 1$. Show that this information determines a unique value of λ (which you should find) for exactly one value of q (which you should also find).

(iii) It is known (instead) that $\mathrm{P}(X = 1 \mid X \leqslant 2) = r$, where r is a given number in the range $0 < r < 1$. Show that this information determines a unique value of λ (which you should find) for exactly one value of r (which you should also find).

Q3, (STEP I, 2010, Q13)

The number of texts that George receives on his mobile phone can be modelled by a Poisson random variable with mean λ texts per hour. Given that the probability George waits between 1 and 2 hours in the morning before he receives his first text is p, show that

$$pe^{2\lambda} - e^{\lambda} + 1 = 0.$$

Given that $4p < 1$, show that there are two positive values of λ that satisfy this equation.

The number of texts that Mildred receives on each of her two mobile phones can be modelled by independent Poisson random variables with different means λ_1 and λ_2 texts per hour. Given that, for each phone, the probability that Mildred waits between 1 and 2 hours in the morning before she receives her first text is also p, find an expression for $\lambda_1 + \lambda_2$ in terms of p.

Find the probability, in terms of p, that she waits between 1 and 2 hours in the morning to receive her first text.

Q4, (STEP II, 2013, Q12)

The random variable U has a Poisson distribution with parameter λ. The random variables X and Y are defined as follows.

$$X = \begin{cases} U & \text{if } U \text{ is } 1, 3, 5, 7, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$Y = \begin{cases} U & \text{if } U \text{ is } 2, 4, 6, 8, \ldots \\ 0 & \text{otherwise} \end{cases}$$

(i) Find $\mathrm{E}(X)$ and $\mathrm{E}(Y)$ in terms of λ, α and β, where

$$\alpha = 1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots \quad \text{and} \quad \beta = \frac{\lambda}{1!} + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \cdots.$$

(ii) Show that

$$\mathrm{Var}\,(X) = \frac{\lambda\alpha + \lambda^2\beta}{\alpha + \beta} - \frac{\lambda^2\alpha^2}{(\alpha+\beta)^2}$$

and obtain the corresponding expression for $\mathrm{Var}\,(Y)$. Are there any non-zero values of λ for which $\mathrm{Var}\,(X) + \mathrm{Var}\,(Y) = \mathrm{Var}\,(X+Y)$?

Q5, (STEP I, 2015, Q12)

The number X of casualties arriving at a hospital each day follows a Poisson distribution with mean 8; that is,

$$P(X = n) = \frac{e^{-8}8^n}{n!}, \quad n = 0, 1, 2, \ldots .$$

Casualties require surgery with probability $\frac{1}{4}$. The number of casualties arriving on any given day is independent of the number arriving on any other day and the casualties require surgery independently of one another.

(i) What is the probability that, on a day when exactly n casualties arrive, exactly r of them require surgery?

(ii) Prove (algebraically) that the number requiring surgery each day also follows a Poisson distribution, and state its mean.

(iii) Given that in a particular randomly chosen week a total of 12 casualties require surgery on Monday and Tuesday, what is the probability that 8 casualties require surgery on Monday? You should give your answer as a fraction in its lowest terms.

Q6, (STEP II, 2017, Q12)

Adam and Eve are catching fish. The number of fish, X, that Adam catches in a fixed time interval T has a Poisson distribution with parameter λ. The number of fish, Y, that Eve catches in the same time interval has a Poisson distribution with parameter μ. The two Poisson variables are independent.

(i) By considering $P(X+Y = r)$, show that the total number of fish caught by Adam and Eve in time T also has a Poisson distribution.

(ii) Given that Adam and Eve catch a total of k fish in time T, where k is fixed, show that the number caught by Adam has a binomial distribution.

(iii) Given that Adam and Eve start fishing at the same time, find the probability that the first fish is caught by Adam.

(iv) You are now given that, for a Poisson distribution with parameter θ, the expected time from any starting point until the next event is θ^{-1}.

Find the expected time from the moment Adam and Eve start fishing until they have each caught at least one fish.
