STEP II - Matrices

STEP II Specification

Matrices

Add, subtract, and multiply conformable matrices; multiply a matrix by a scalar.

Understand and use zero and identity matrices.

Use matrices to represent linear transformations in 2-D; successive transformations; single transformations in 3-D (3-D transformations confined to reflection in one of $x = 0$, $y = 0$, $z = 0$ or rotation about one of the coordinate axes). (Knowledge of 3-D vectors is assumed.)

Find invariant points and lines for a linear transformation.

Calculate determinants of 2×2 matrices ***and interpret as scale factors, including the effect on orientation***.

Understand and use singular and non-singular matrices; properties of inverse matrices.

Calculate and use the inverse of a non-singular 2×2 matrix.

Q1, (STEP III, 1998, Q5)

The exponential of a square matrix $\mathbf{A}$ is defined to be

$$\exp(\mathbf{A}) = \sum_{r=0}^{\infty} \frac{1}{r!}\mathbf{A}^r ,$$

where $\mathbf{A}^0 = \mathbf{I}$ and $\mathbf{I}$ is the identity matrix.

Let

$$\mathbf{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Show that $\mathbf{M}^2 = -\mathbf{I}$ and hence express $\exp(\theta\mathbf{M})$ as a single 2×2 matrix, where θ is a real number. Explain the geometrical significance of $\exp(\theta\mathbf{M})$.

Let

$$\mathbf{N} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

Express similarly $\exp(s\mathbf{N})$, where s is a real number, and explain the geometrical significance of $\exp(s\mathbf{N})$.

For which values of θ does

$$\exp(s\mathbf{N})\ \exp(\theta\mathbf{M}) = \exp(\theta\mathbf{M})\ \exp(s\mathbf{N})$$

for all s? Interpret this fact geometrically.

Q2, (STEP III, 2000, Q5)

Given two non-zero vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ we define $\Delta(\mathbf{a}, \mathbf{b})$ by $\Delta(\mathbf{a}, \mathbf{b}) = a_1 b_2 - a_2 b_1$.

Let A, B and C be points with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively, no two of which are parallel. Let P, Q and R be points with position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$, respectively, none of which are parallel.

(i) Show that there exists a 2×2 matrix $\mathbf{M}$ such that P and Q are the images of A and B under the transformation represented by $\mathbf{M}$.

(ii) Show that $\Delta(\mathbf{a}, \mathbf{b})\, \mathbf{c} + \Delta(\mathbf{c}, \mathbf{a})\, \mathbf{b} + \Delta(\mathbf{b}, \mathbf{c})\, \mathbf{a} = \mathbf{0}$.

Hence, or otherwise, prove that a necessary and sufficient condition for the points P, Q, and R to be the images of points A, B and C under the transformation represented by some 2×2 matrix $\mathbf{M}$ is that

$$\Delta(\mathbf{a}, \mathbf{b}) : \Delta(\mathbf{b}, \mathbf{c}) : \Delta(\mathbf{c}, \mathbf{a}) = \Delta(\mathbf{p}, \mathbf{q}) : \Delta(\mathbf{q}, \mathbf{r}) : \Delta(\mathbf{r}, \mathbf{p}) .$$

Q3, (STEP II, 2019, Q8)

The domain of the function f is the set of all 2×2 matrices and its range is the set of real numbers. Thus, if $\mathbf{M}$ is a 2×2 matrix, then $\mathrm{f}(\mathbf{M}) \in \mathbb{R}$.

The function f has the property that $\mathrm{f}(\mathbf{MN}) = \mathrm{f}(\mathbf{M})\mathrm{f}(\mathbf{N})$ for any 2×2 matrices $\mathbf{M}$ and $\mathbf{N}$.

(i) You are given that there is a matrix $\mathbf{M}$ such that $\mathrm{f}(\mathbf{M}) \neq 0$. Let $\mathbf{I}$ be the 2×2 identity matrix. By considering $\mathrm{f}(\mathbf{MI})$, show that $\mathrm{f}(\mathbf{I}) = 1$.

(ii) Let $\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. You are given that $\mathrm{f}(\mathbf{J}) \neq 1$. By considering $\mathbf{J}^2$, evaluate $\mathrm{f}(\mathbf{J})$.

Using $\mathbf{J}$, show that, for any real numbers a, b, c and d,

$$\mathrm{f}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = -\mathrm{f}\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix}\right) = \mathrm{f}\left(\begin{pmatrix} d & c \\ b & a \end{pmatrix}\right).$$

(iii) Let $\mathbf{K} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ where $k \in \mathbb{R}$. Use $\mathbf{K}$ to show that, if the second row of the matrix $\mathbf{A}$ is a multiple of the first row, then $\mathrm{f}(\mathbf{A}) = 0$.

(iv) Let $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. By considering the matrices $\mathbf{P}^2$, $\mathbf{P}^{-1}$, and $\mathbf{K}^{-1}\mathbf{PK}$ for suitable values of k, evaluate $\mathrm{f}(\mathbf{P})$.

Q4, (STEP III, 2019, Q3)

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(i) You are given that the transformation represented by $\mathbf{A}$ has a line L_1 of invariant points (so that each point on L_1 is transformed to itself). Let (x, y) be a point on L_1. Show that $\big((a-1)(d-1) - bc\big)xy = 0$.

Show further that $(a-1)(d-1) = bc$.

What can be said about $\mathbf{A}$ if L_1 does not pass through the origin?

(ii) By considering the cases $b \neq 0$ and $b = 0$ separately, show that if $(a-1)(d-1) = bc$ then the transformation represented by $\mathbf{A}$ has a line of invariant points. You should identify the line in the different cases that arise.

(iii) You are given instead that the transformation represented by $\mathbf{A}$ has an invariant line L_2 (so that each point on L_2 is transformed to a point on L_2) and that L_2 does not pass through the origin. If L_2 has the form $y = mx + k$, show that $(a-1)(d-1) = bc$.