STEP 2 – Integration with Trigonometric Functions

Further calculus

Integrate functions of the form $(1+x^2)^{-1}$ and $(1-x^2)^{-\frac{1}{2}}$ and be able to choose trigonometric substitutions to integrate associated functions.

Q1, (STEP III, 2006, Q2)

Let

$$I = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2\theta}{1-\sin\theta\sin 2\alpha}\,\mathrm{d}\theta \quad \text{and} \quad J = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sec^2\theta}{1+\tan^2\theta\cos^2 2\alpha}\,\mathrm{d}\theta$$

where $0 < \alpha < \frac{1}{4}\pi$.

(i) Show that $I = \displaystyle\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2\theta}{1+\sin\theta\sin 2\alpha}\,\mathrm{d}\theta$ and hence that $2I = \displaystyle\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{2}{1+\tan^2\theta\cos^2 2\alpha}\,\mathrm{d}\theta$.

(ii) Find J.

(iii) By considering $I\sin^2 2\alpha + J\cos^2 2\alpha$, or otherwise, show that $I = \frac{1}{2}\pi\sec^2\alpha$.

(iv) Evaluate I in the case $\frac{1}{4}\pi < \alpha < \frac{1}{2}\pi$.

Q2, (STEP III, 2011, Q4)

The following result applies to any function f which is continuous, has positive gradient and satisfies $\mathrm{f}(0) = 0$:

$$ab \leqslant \int_0^a \mathrm{f}(x)\,\mathrm{d}x + \int_0^b \mathrm{f}^{-1}(y)\,\mathrm{d}y\,, \qquad (*)$$

where f^{-1} denotes the inverse function of f, and $a \geqslant 0$ and $b \geqslant 0$.

(i) By considering the graph of $y = \mathrm{f}(x)$, explain briefly why the inequality $(*)$ holds.

In the case $a > 0$ and $b > 0$, state a condition on a and b under which equality holds.

(ii) By taking $\mathrm{f}(x) = x^{p-1}$ in $(*)$, where $p > 1$, show that if $\dfrac{1}{p} + \dfrac{1}{q} = 1$ then

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}\,.$$

Verify that equality holds under the condition you stated above.

(iii) Show that, for $0 \leqslant a \leqslant \frac{1}{2}\pi$ and $0 \leqslant b \leqslant 1$,

$$ab \leqslant b\arcsin b + \sqrt{1-b^2} - \cos a\,.$$

Deduce that, for $t \geqslant 1$,

$$\arcsin(t^{-1}) \geqslant t - \sqrt{t^2-1}\,.$$

Q3, (STEP III, 2017, Q6)

In this question, you are not permitted to use any properties of trigonometric functions or inverse trigonometric functions.

The function T is defined for $x > 0$ by

$$\mathrm{T}(x) = \int_0^x \frac{1}{1+u^2}\,\mathrm{d}u\,,$$

and $T_\infty = \displaystyle\int_0^\infty \frac{1}{1+u^2}\,\mathrm{d}u$ (which has a finite value).

(i) By making an appropriate substitution in the integral for $\mathrm{T}(x)$, show that

$$\mathrm{T}(x) = \mathrm{T}_\infty - \mathrm{T}(x^{-1})\,.$$

(ii) Let $v = \dfrac{u+a}{1-au}$, where a is a constant. Verify that, for $u \neq a^{-1}$,

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{1+v^2}{1+u^2}\,.$$

Hence show that, for $a > 0$ and $x < \dfrac{1}{a}$,

$$\mathrm{T}(x) = \mathrm{T}\left(\frac{x+a}{1-ax}\right) - \mathrm{T}(a)\,.$$

Deduce that

$$\mathrm{T}(x^{-1}) = 2\mathrm{T}_\infty - \mathrm{T}\left(\frac{x+a}{1-ax}\right) - \mathrm{T}(a^{-1})$$

and hence that, for $b > 0$ and $y > \dfrac{1}{b}$,

$$\mathrm{T}(y) = 2\mathrm{T}_\infty - \mathrm{T}\left(\frac{y+b}{by-1}\right) - \mathrm{T}(b)\,.$$

(iii) Use the above results to show that $\mathrm{T}(\sqrt{3}) = \frac{2}{3}\mathrm{T}_\infty$ and $\mathrm{T}(\sqrt{2}-1) = \frac{1}{4}\mathrm{T}_\infty$.

Q4, (STEP III, 2014, Q4)

(i) Let

$$I = \int_0^1 \left((y')^2 - y^2\right) \mathrm{d}x \qquad \text{and} \qquad I_1 = \int_0^1 (y' + y\tan x)^2 \, \mathrm{d}x\,,$$

where y is a given function of x satisfying $y = 0$ at $x = 1$. Show that $I - I_1 = 0$ and deduce that $I \geqslant 0$. Show further that $I = 0$ only if $y = 0$ for all x $(0 \leqslant x \leqslant 1)$.

(ii) Let

$$J = \int_0^1 \left((y')^2 - a^2y^2\right) \mathrm{d}x\,,$$

where a is a given positive constant and y is a given function of x, not identically zero, satisfying $y = 0$ at $x = 1$. By considering an integral of the form

$$\int_0^1 (y' + ay\tan bx)^2 \, \mathrm{d}x\,,$$

where b is suitably chosen, show that $J \geqslant 0$. You should state the range of values of a, in the form $a < k$, for which your proof is valid.

In the case $a = k$, find a function y (not everywhere zero) such that $J = 0$.

Q5, (STEP I, 2004, Q4)

Differentiate $\sec t$ with respect to t.

(i) Use the substitution $x = \sec t$ to show that $\displaystyle\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}} \, \mathrm{d}x = \frac{\sqrt{3}-2}{8} + \frac{\pi}{24}$.

(ii) Determine $\displaystyle\int \frac{1}{(x+2)\sqrt{(x+1)(x+3)}} \, \mathrm{d}x$.

(iii) Determine $\displaystyle\int \frac{1}{(x+2)\sqrt{x^2+4x-5}} \, \mathrm{d}x$.

Q6, (STEP I, 2009, Q7)

Show that, for any integer m,

$$\int_0^{2\pi} \mathrm{e}^x \cos mx \, \mathrm{d}x = \frac{1}{m^2+1}\left(\mathrm{e}^{2\pi} - 1\right).$$

(i) Expand $\cos(A+B) + \cos(A-B)$. Hence show that

$$\int_0^{2\pi} \mathrm{e}^x \cos x \cos 6x \, \mathrm{d}x = \tfrac{19}{650}\left(\mathrm{e}^{2\pi} - 1\right).$$

(ii) Evaluate $\displaystyle\int_0^{2\pi} \mathrm{e}^x \sin 2x \sin 4x \cos x \, \mathrm{d}x$.
