STEP II - Ellipses and Hyperbolae

STEP II Specification

Further algebra and functions

Sketch curves of the form $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$; find equations of their asymptotes where appropriate.

Q1, (STEP III, 2008, Q3)

The point $P(a\cos\theta\,,\, b\sin\theta)$, where $a > b > 0$, lies on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.$$

The point $S(-ea\,,\,0)$, where $b^2 = a^2(1-e^2)$, is a focus of the ellipse. The point N is the foot of the perpendicular from the origin, O, to the tangent to the ellipse at P. The lines SP and ON intersect at T. Show that the y-coordinate of T is

$$\frac{b\sin\theta}{1+e\cos\theta}\,.$$

Show that T lies on the circle with centre S and radius a.

Q2, (STEP III 2017, Q7)

Show that the point T with coordinates

$$\left(\frac{a(1-t^2)}{1+t^2}\,,\,\frac{2bt}{1+t^2}\right) \qquad (*)$$

(where a and b are non-zero) lies on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.$$

(i) The line L is the tangent to the ellipse at T. The point (X,Y) lies on L, and $X^2 \neq a^2$. Show that

$$(a+X)bt^2 - 2aYt + b(a-X) = 0\,.$$

Deduce that if $a^2Y^2 > (a^2 - X^2)b^2$, then there are two distinct lines through (X,Y) that are tangents to the ellipse. Interpret this result geometrically. Show, by means of a sketch, that the result holds also if $X^2 = a^2$.

(ii) The distinct points P and Q are given by $(*)$, with $t = p$ and $t = q$, respectively. The tangents to the ellipse at P and Q meet at the point with coordinates (X,Y), where $X^2 \neq a^2$. Show that

$$(a+X)pq = a - X$$

and find an expression for $p+q$ in terms of a, b, X and Y.

Given that the tangents meet the y-axis at points $(0,y_1)$ and $(0,y_2)$, where $y_1 + y_2 = 2b$, show that

$$\frac{X^2}{a^2} + \frac{Y}{b} = 1\,.$$

Q3, (STEP III, 2018, Q4)

The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,,$$

where $a > b > 0$. Show that the equation of the tangent to the hyperbola at P can be written as

$$bx - ay\sin\theta = ab\cos\theta\,.$$

(i) This tangent meets the lines $\dfrac{x}{a} = \dfrac{y}{b}$ and $\dfrac{x}{a} = -\dfrac{y}{b}$ at S and T, respectively.

How is the mid-point of ST related to P?

(ii) The point $Q(a\sec\phi, b\tan\phi)$ also lies on the hyperbola and the tangents to the hyperbola at P and Q are perpendicular. These two tangents intersect at (x, y).

Obtain expressions for x^2 and y^2 in terms of a, θ and ϕ.

Hence, or otherwise, show that $x^2 + y^2 = a^2 - b^2$.