STEP II – Continuous Random Variables

Probability distributions

Understand and use the mathematics of continuous probability density functions and cumulative distribution functions; including finding probabilities and the calculation of mean, variance, median, mode, and expectation by explicit integration for a given (possibly unfamiliar) distribution; the notation $\mathbf{f}(x) = \mathbf{F}'(x)$.

Q1, (STEP II, 2009, Q12)

A continuous random variable X has probability density function given by

$$\mathrm{f}(x) = \begin{cases} 0 & \text{for } x < 0 \\ k\mathrm{e}^{-2x^2} & \text{for } 0 \leqslant x < \infty \,, \end{cases}$$

where k is a constant.

(i) Sketch the graph of $\mathrm{f}(x)$.

(ii) Find the value of k.

(iii) Determine $\mathrm{E}(X)$ and $\mathrm{Var}\,(X)$.

(iv) Use statistical tables to find, to three significant figures, the median value of X.

Q2, (STEP II, 2010, Q12)

The continuous random variable X has probability density function $\mathrm{f}(x)$, where

$$\mathrm{f}(x) = \begin{cases} a & \text{for } 0 \leqslant x < k \\ b & \text{for } k \leqslant x \leqslant 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $a > b > 0$ and $0 < k < 1$. Show that $a > 1$ and $b < 1$.

(i) Show that

$$\mathrm{E}(X) = \frac{1 - 2b + ab}{2(a-b)} \,.$$

(ii) Show that the median, M, of X is given by $M = \dfrac{1}{2a}$ if $a + b \geqslant 2ab$ and obtain an expression for the median if $a + b \leqslant 2ab$.

(iii) Show that $M < \mathrm{E}(X)$.

Q3, (STEP II, 2014, Q12)

The lifetime of a fly (measured in hours) is given by the continuous random variable T with probability density function $\mathrm{f}(t)$ and cumulative distribution function $\mathrm{F}(t)$. The *hazard function*, $\mathrm{h}(t)$, is defined, for $\mathrm{F}(t) < 1$, by

$$\mathrm{h}(t) = \frac{\mathrm{f}(t)}{1 - \mathrm{F}(t)}.$$

(i) Given that the fly lives to at least time t, show that the probability of its dying within the following δt is approximately $\mathrm{h}(t)\,\delta t$ for small values of δt.

(ii) Find the hazard function in the case $\mathrm{F}(t) = t/a$ for $0 < t < a$. Sketch $\mathrm{f}(t)$ and $\mathrm{h}(t)$ in this case.

(iii) The random variable T is distributed on the interval $t > a$, where $a > 0$, and its hazard function is t^{-1}. Determine the probability density function for T.

(iv) Show that $\mathrm{h}(t)$ is constant for $t > b$ and zero otherwise if and only if $\mathrm{f}(t) = k\mathrm{e}^{-k(t-b)}$ for $t > b$, where k is a positive constant.

(v) The random variable T is distributed on the interval $t > 0$ and its hazard function is given by

$$\mathrm{h}(t) = \left(\frac{\lambda}{\theta^\lambda}\right) t^{\lambda - 1},$$

where λ and θ are positive constants. Find the probability density function for T.

Q4, (STEP II, 2006, Q14)

Sketch the graph of $y = \dfrac{1}{x \ln x}$ for $x > 0$, $x \neq 1$. You may assume that $x \ln x \to 0$ as $x \to 0$.

The continuous random variable X has probability density function

$$\mathrm{f}(x) = \begin{cases} \dfrac{\lambda}{x \ln x} & \text{for } a \leqslant x \leqslant b\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where a, b and λ are suitably chosen constants.

(i) In the case $a = 1/4$ and $b = 1/2$, find λ.

(ii) In the case $\lambda = 1$ and $a > 1$, show that $b = a^{\mathrm{e}}$.

iii) In the case $\lambda = 1$ and $a = \mathrm{e}$, show that $\mathrm{P}(\mathrm{e}^{3/2} \leqslant X \leqslant \mathrm{e}^2) \approx \frac{31}{108}$.

iv) In the case $\lambda = 1$ and $a = \mathrm{e}^{1/2}$, find $\mathrm{P}(\mathrm{e}^{3/2} \leqslant X \leqslant \mathrm{e}^2)$.

Q5, (STEP II, 2015, Q13)

The maximum height X of flood water each year on a certain river is a random variable with probability density function f given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geqslant 0\,, \\ 0 & \text{otherwise,} \end{cases}$$

where λ is a positive constant.

It costs ky pounds each year to prepare for flood water of height y or less, where k is a positive constant and $y \geqslant 0$. If $X \leqslant y$ no further costs are incurred but if $X > y$ the additional cost of flood damage is $a(X - y)$ pounds where a is a positive constant.

(i) Let C be the total cost of dealing with the floods in the year. Show that the expectation of C is given by

$$E(C) = ky + \frac{a}{\lambda}e^{-\lambda y}\,.$$

How should y be chosen in order to minimise $E(C)$, in the different cases that arise according to the value of a/k?

(ii) Find the variance of C, and show that the more that is spent on preparing for flood water in advance the smaller this variance.

Q6, (STEP II, 2007, Q14)

The random variable X has a continuous probability density function $f(x)$ given by

$$f(x) = \begin{cases} 0 & \text{for } x \leqslant 1 \\ \ln x & \text{for } 1 \leqslant x \leqslant k \\ \ln k & \text{for } k \leqslant x \leqslant 2k \\ a - bx & \text{for } 2k \leqslant x \leqslant 4k \\ 0 & \text{for } x \geqslant 4k \end{cases}$$

where k, a and b are constants.

(i) Sketch the graph of $y = f(x)$.

(ii) Determine a and b in terms of k and find the numerical values of k, a and b.

(iii) Find the median value of X.

Q7, (STEP I, Jun 2011, Q13)

In this question, you may use without proof the following result:

$$\int \sqrt{4-x^2}\,dx = 2\arcsin(\tfrac{1}{2}x) + \tfrac{1}{2}x\sqrt{4-x^2} + c\,.$$

A random variable X has probability density function f given by

$$f(x) = \begin{cases} 2k & -a \leqslant x < 0 \\ k\sqrt{4-x^2} & 0 \leqslant x \leqslant 2 \\ 0 & \text{otherwise,} \end{cases}$$

where k and a are positive constants.

(i) Find, in terms of a, the mean of X.

(ii) Let d be the value of X such that $P(X > d) = \frac{1}{10}$. Show that $d < 0$ if $2a > 9\pi$ and find an expression for d in terms of a in this case.

(iii) Given that $d = \sqrt{2}$, find a.

Q8, (STEP I, 2012, Q12)

Fire extinguishers may become faulty at any time after manufacture and are tested annually on the anniversary of manufacture.

The time T years after manufacture until a fire extinguisher becomes faulty is modelled by the continuous probability density function

$$f(t) = \begin{cases} \dfrac{2t}{(1+t^2)^2} & \text{for } t \geqslant 0\,, \\ 0 & \text{otherwise.} \end{cases}$$

A faulty fire extinguisher will fail an annual test with probability p, in which case it is destroyed immediately. A non-faulty fire extinguisher will always pass the test. All of the annual tests are independent.

Show that the probability that a randomly chosen fire extinguisher will be destroyed exactly three years after its manufacture is $p(5p^2 - 13p + 9)/10$.

Find the probability that a randomly chosen fire extinguisher that was destroyed exactly three years after its manufacture was faulty 18 months after its manufacture.
