STEP II – Complex Numbers

STEP II Specification

Complex numbers

Solve any quadratic equation with real ***or complex*** coefficients; solve cubic or quartic equations with real ***or complex*** coefficients (given sufficient information to deduce at least one root for cubics or at least one complex root or quadratic factor for quartics).

Add, subtract, multiply, and divide complex numbers in the form $x + iy$ with x and y real; understand and use the terms 'real part' and 'imaginary part'.

Understand and use the complex conjugate; know that non-real roots of polynomial equations with real coefficients occur in conjugate pairs.

Use and interpret Argand diagrams.

Convert between the Cartesian form and the modulus-argument form of a complex number (knowledge of radians is assumed).

Multiply and divide complex numbers in modulus-argument form (knowledge of radians and compound angle formulae is assumed).

Construct and interpret simple loci in the Argand diagram such as, ***but not limited to***, $|z - a| = r$ and $\arg(z - a) = \theta$ (knowledge of radians is assumed).

Q1, (STEP III, 2018, Q6)

(i) The distinct points A, Q and C lie on a straight line in the Argand diagram, and represent the distinct complex numbers a, q and c, respectively. Show that $\dfrac{q-a}{c-a}$ is real and hence that $(c-a)(q^*-a^*) = (c^*-a^*)(q-a)$.

Given that $aa^* = cc^* = 1$, show further that

$$q + acq^* = a + c\,.$$

(ii) The distinct points A, B, C and D lie, in anticlockwise order, on the circle of unit radius with centre at the origin (so that, for example, $aa^* = 1$). The lines AC and BD meet at Q. Show that

$$(ac - bd)q^* = (a+c) - (b+d)\,,$$

where b and d are complex numbers represented by the points B and D respectively, and show further that

$$(ac-bd)(q+q^*) = (a-b)(1+cd) + (c-d)(1+ab)\,.$$

(iii) The lines AB and CD meet at P, which represents the complex number p. Given that p is real, show that $p(1+ab) = a+b$. Given further that $ac - bd \neq 0$, show that

$$p(q+q^*) = 2\,.$$

Q2, (STEP III, 2007, Q6)

The distinct points P, Q, R and S in the Argand diagram lie on a circle of radius a centred at the origin and are represented by the complex numbers p, q, r and s, respectively. Show that

$$pq = -a^2 \frac{p-q}{p^* - q^*}.$$

Deduce that, if the chords PQ and RS are perpendicular, then $pq + rs = 0$.

The distinct points $A_1, A_2, \ldots, A_n$ (where $n \geqslant 3$) lie on a circle. The points $B_1, B_2, \ldots, B_n$ lie on the same circle and are chosen so that the chords $B_1B_2, B_2B_3, \ldots, B_nB_1$ are perpendicular, respectively, to the chords $A_1A_2, A_2A_3, \ldots, A_nA_1$. Show that, for $n = 3$, there are only two choices of B_1 for which this is possible. What is the corresponding result for $n = 4$? State the corresponding results for values of n greater than 4.

Q3, (STEP III, 2015, Q6)

(i) Let w and z be complex numbers, and let $u = w + z$ and $v = w^2 + z^2$. Prove that w and z are real if and only if u and v are real and $u^2 \leqslant 2v$.

(ii) The complex numbers u, w and z satisfy the equations

$$\begin{aligned} w + z - u &= 0 \\ w^2 + z^2 - u^2 &= -\tfrac{2}{3} \\ w^3 + z^3 - \lambda u &= -\lambda \end{aligned}$$

where λ is a positive real number. Show that for all values of λ except one (which you should find) there are three possible values of u, all real.

Are w and z necessarily real? Give a proof or counterexample.

Q4, (STEP III, 2008, Q7)

The points A, B and C in the Argand diagram are the vertices of an equilateral triangle described anticlockwise. Show that the complex numbers a, b and c representing A, B and C satisfy

$$2c = (a + b) + i\sqrt{3}(b - a).$$

Find a similar relation in the case that A, B and C are the vertices of an equilateral triangle described clockwise.

(i) The quadrilateral $DEFG$ lies in the Argand diagram. Show that points P, Q, R and S can be chosen so that PDE, QEF, RFG and SGD are equilateral triangles and $PQRS$ is a parallelogram.

(ii) The triangle LMN lies in the Argand diagram. Show that the centroids U, V and W of the equilateral triangles drawn externally on the sides of LMN are the vertices of an equilateral triangle.

[**Note:** The *centroid* of a triangle with vertices represented by the complex numbers x, y and z is the point represented by $\frac{1}{3}(x + y + z)$.]

Q5, (STEP III, 2012, Q6)

Let $x + iy$ be a root of the quadratic equation $z^2 + pz + 1 = 0$, where p is a real number. Show that $x^2 - y^2 + px + 1 = 0$ and $(2x + p)y = 0$. Show further that

$$\text{either } p = -2x \quad \text{or} \quad p = -(x^2 + 1)/x \text{ with } x \neq 0 \,.$$

Hence show that the set of points in the Argand diagram that can (as p varies) represent roots of the quadratic equation consists of the real axis with one point missing and a circle. This set of points is called the *root locus* of the quadratic equation.

Obtain and sketch in the Argand diagram the root locus of the equation

$$pz^2 + z + 1 = 0$$

and the root locus of the equation

$$pz^2 + p^2 z + 2 = 0 \,.$$

Q6, (STEP III, 2013, Q6)

Let z and w be complex numbers. Use a diagram to show that $|z - w| \leqslant |z| + |w|$.

For any complex numbers z and w, E is defined by

$$E = zw^* + z^*w + 2|zw| \,.$$

(i) Show that $|z - w|^2 = (|z| + |w|)^2 - E$, and deduce that E is real and non-negative.

(ii) Show that $|1 - zw^*|^2 = (1 + |zw|)^2 - E$.

Hence show that, if both $|z| > 1$ and $|w| > 1$, then

$$\frac{|z - w|}{|1 - zw^*|} \leqslant \frac{|z| + |w|}{1 + |zw|} \,.$$

Does this inequality also hold if both $|z| < 1$ and $|w| < 1$?

Q7, (STEP III, 2014, Q5)

A quadrilateral drawn in the complex plane has vertices A, B, C and D, labelled anticlockwise. These vertices are represented, respectively, by the complex numbers a, b, c and d. Show that $ABCD$ is a parallelogram (defined as a quadrilateral in which opposite sides are parallel and equal in length) if and only if $a + c = b + d$. Show further that, in this case, $ABCD$ is a square if and only if $\mathrm{i}(a - c) = b - d$.

Let $PQRS$ be a quadrilateral in the complex plane, with vertices labelled anticlockwise, the internal angles of which are all less than 180°. Squares with centres X, Y, Z and T are constructed externally to the quadrilateral on the sides PQ, QR, RS and SP, respectively.

(i) If P and Q are represented by the complex numbers p and q, respectively, show that X can be represented by

$$\tfrac{1}{2}\big(p(1 + \mathrm{i}) + q(1 - \mathrm{i})\big) \,.$$

(ii) Show that $XYZT$ is a square if and only if $PQRS$ is a parallelogram.

Q8, (STEP III, 2014, Q5)

A quadrilateral drawn in the complex plane has vertices A, B, C and D, labelled anticlockwise. These vertices are represented, respectively, by the complex numbers a, b, c and d. Show that $ABCD$ is a parallelogram (defined as a quadrilateral in which opposite sides are parallel and equal in length) if and only if $a + c = b + d$. Show further that, in this case, $ABCD$ is a square if and only if $\mathrm{i}(a - c) = b - d$.

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