STEP I – The Normal Distribution

Statistical distributions

Understand and use the Normal distribution; find probabilities using the Normal distribution; ***convert to the standard Normal distribution by translation and scaling.***

Statistical hypothesis testing

Conduct a statistical hypothesis test for the mean of a Normal distribution with known, given, or assumed variance and interpret the results in context.

Q1, (STEP III, 2005, Q14)

In this question, $\Phi(z)$ is the cumulative distribution function of a standard normal random variable.

A random variable is known to have a Normal distribution with mean μ and standard deviation either σ_0 or σ_1, where $\sigma_0 < \sigma_1$. The mean, $\overline{X}$, of a random sample of n values of X is to be used to test the hypothesis $\mathrm{H}_0 : \sigma = \sigma_0$ against the alternative $\mathrm{H}_1 : \sigma = \sigma_1$.

Explain carefully why it is appropriate to use a two sided test of the form: accept H_0 if $\mu - c < \overline{X} < \mu + c$, otherwise accept H_1.

Given that the probability of accepting H_1 when H_0 is true is α, determine c in terms of n, σ_0 and z_α, where z_α is defined by $\Phi(z_\alpha) = 1 - \dfrac{\alpha}{2}$.

The probability of accepting H_0 when H_1 is true is denoted by β. Show that β is independent of n.

Given that $\Phi(1.960) \approx 0.975$ and that $\Phi(0.063) \approx 0.525$, determine, approximately, the minimum value of $\dfrac{\sigma_1}{\sigma_0}$ if α and β are both to be less than 0.05.

Q2, (STEP III, 2006, Q12)

***Technically, this question is of the difficulty you could expect on STEP II or III, but the topic of normal distribution is examinable on STEP I.**

Fifty times a year, 1024 tourists disembark from a cruise liner at a port. From there they must travel to the city centre either by bus or by taxi. Tourists are equally likely to be directed to the bus station or to the taxi rank. Each bus of the bus company holds 32 passengers, and the company currently runs 15 buses. The company makes a profit of £1 for each passenger carried. It carries as many passengers as it can, with any excess being (eventually) transported by taxi. Show that the largest annual licence fee, in pounds, that the company should consider paying to be allowed to run an extra bus is approximately

$$1600\Phi(2) - \frac{800}{\sqrt{2\pi}}\left(1 - \mathrm{e}^{-2}\right),$$

where $\Phi(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} \mathrm{e}^{-\frac{1}{2}t^2}\,\mathrm{d}t$.

[*You should not consider continuity corrections.*]

Q3, (2012, Q13)

(i) The random variable Z has a Normal distribution with mean 0 and variance 1. Show that the expectation of Z given that $a < Z < b$ is

$$\frac{\exp(-\frac{1}{2}a^2) - \exp(-\frac{1}{2}b^2)}{\sqrt{2\pi}\,\big(\Phi(b) - \Phi(a)\big)},$$

where Φ denotes the cumulative distribution function for Z.

(ii) The random variable X has a Normal distribution with mean μ and variance σ^2. Show that

$$\mathrm{E}(X \mid X > 0) = \mu + \sigma \mathrm{E}(Z \mid Z > -\mu/\sigma).$$

Hence, or otherwise, show that the expectation, m, of $|X|$ is given by

$$m = \mu\big(1 - 2\Phi(-\mu/\sigma)\big) + \sigma\sqrt{2/\pi}\,\exp(-\tfrac{1}{2}\mu^2/\sigma^2)\,.$$

Obtain an expression for the variance of $|X|$ in terms of μ, σ and m.