STEP I – Numerical Reasoning and Proof

Proof

Understand and use the structure of mathematical proof, proceeding from given assumptions through a series of logical steps to a conclusion; use methods of proof, including proof by deduction, proof by exhaustion, ***proof by induction***.

Understand and use the terms 'necessary and sufficient' and 'if and only if'.

Disproof by counter-example.

Proof by contradiction (including proof of the irrationality of $\sqrt{2}$ and the infinity of primes, and application to unfamiliar proofs).

Q1, (STEP I, 2005, Q1)

47231 is a five-digit number whose digits sum to $4 + 7 + 2 + 3 + 1 = 17$.

(i) Show that there are 15 five-digit numbers whose digits sum to 43. You should explain your reasoning clearly.

(ii) How many five-digit numbers are there whose digits sum to 39?

Q2, (STEP I, 2006, Q1)

Find the integer, n, that satisfies $n^2 < 33127 < (n+1)^2$. Find also a small integer m such that $(n+m)^2 - 33127$ is a perfect square. Hence express 33127 in the form pq, where p and q are integers greater than 1.

By considering the possible factorisations of 33127, show that there are exactly two values of m for which $(n+m)^2 - 33127$ is a perfect square, and find the other value.

Q3, (STEP I, 2014, Q1)

All numbers referred to in this question are non-negative integers.

(i) Express each of the numbers 3, 5, 8, 12 and 16 as the difference of two non-zero squares.

(ii) Prove that any odd number can be written as the difference of two squares.

(iii) Prove that all numbers of the form $4k$, where k is a non-negative integer, can be written as the difference of two squares.

(iv) Prove that no number of the form $4k + 2$, where k is a non-negative integer, can be written as the difference of two squares.

(v) Prove that any number of the form pq, where p and q are prime numbers greater than 2, can be written as the difference of two squares in exactly two distinct ways. Does this result hold if p is a prime greater than 2 and $q = 2$?

(vi) Determine the number of distinct ways in which 675 can be written as the difference of two squares.

Q4, (STEP I, 2007, Q1)

A positive integer with $2n$ digits (the first of which must not be 0) is called a *balanced number* if the sum of the first n digits equals the sum of the last n digits. For example, 1634 is a 4-digit balanced number, but 123401 is not a balanced number.

(i) Show that seventy 4-digit balanced numbers can be made using the digits 0, 1, 2, 3 and 4.

(ii) Show that $\frac{1}{6}k(k+1)(4k+5)$ 4-digit balanced numbers can be made using the digits 0 to k.

You may use the identity $\sum_{r=0}^{n} r^2 \equiv \frac{1}{6}n(n+1)(2n+1)$.

Q5, (STEP I, 2009, Q1)

A *proper factor* of an integer N is a positive integer, not 1 or N, that divides N.

(i) Show that $3^2 \times 5^3$ has exactly 10 proper factors. Determine how many other integers of the form $3^m \times 5^n$ (where m and n are integers) have exactly 10 proper factors.

(ii) Let N be the smallest positive integer that has exactly 426 proper factors. Determine N, giving your answer in terms of its prime factors.
