STEP I - Integration 2

Q1, (STEP I, 2014, Q2)

(i) Show that $\int \ln(2-x)\,\mathrm{d}x = -(2-x)\ln(2-x) + (2-x) + c$, where $x < 2$.

(ii) Sketch the curve A given by $y = \ln|x^2 - 4|$.

(iii) Show that the area of the finite region enclosed by the positive x-axis, the y-axis and the curve A is $4\ln(2+\sqrt{3}) - 2\sqrt{3}$.

(iv) The curve B is given by $y = \big|\ln|x^2 - 4|\big|$. Find the area between the curve B and the x-axis with $|x| < 2$.

[*Note: you may assume that* $t\ln t \to 0$ *as* $t \to 0$.]

Q2, (STEP I, 2015, Q5)

(i) The function f is defined, for $x > 0$, by

$$\mathrm{f}(x) = \int_1^3 (t-1)^{x-1}\,\mathrm{d}t\,.$$

By evaluating the integral, sketch the curve $y = \mathrm{f}(x)$.

(ii) The function g is defined, for $-\infty < x < \infty$, by

$$\mathrm{g}(x) = \int_{-1}^{1} \frac{1}{\sqrt{1-2xt+x^2}}\,\mathrm{d}t\,.$$

By evaluating the integral, sketch the curve $y = \mathrm{g}(x)$.

Q3, (STEP I, 2017, Q1)

(i) Use the substitution $u = x\sin x + \cos x$ to find

$$\int \frac{x}{x\tan x + 1}\,\mathrm{d}x\,.$$

Find by means of a similar substitution, or otherwise,

$$\int \frac{x}{x\cot x - 1}\,\mathrm{d}x\,.$$

(ii) Use a substitution to find

$$\int \frac{x\sec^2 x\,\tan x}{x\sec^2 x - \tan x}\,\mathrm{d}x$$

and

$$\int \frac{x\sin x\cos x}{(x - \sin x\cos x)^2}\,\mathrm{d}x\,.$$

Q4, (STEP I, 2018, Q8)

The functions s and c satisfy $\mathrm{s}(0) = 0$, $\mathrm{c}(0) = 1$ and

$$\mathrm{s}'(x) = \mathrm{c}(x)^2,$$

$$\mathrm{c}'(x) = -\mathrm{s}(x)^2.$$

You may assume that s and c are uniquely defined by these conditions.

(i) Show that $\mathrm{s}(x)^3 + \mathrm{c}(x)^3$ is constant, and deduce that $\mathrm{s}(x)^3 + \mathrm{c}(x)^3 = 1$.

(ii) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\mathrm{s}(x)\mathrm{c}(x)\Big) = 2\mathrm{c}(x)^3 - 1$$

and find (and simplify) an expression in terms of $\mathrm{c}(x)$ for $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{s}(x)}{\mathrm{c}(x)}\right)$.

(iii) Find the integrals

$$\int \mathrm{s}(x)^2\,\mathrm{d}x \qquad \text{and} \qquad \int \mathrm{s}(x)^5\,\mathrm{d}x.$$

(iv) Given that s has an inverse function, s^{-1}, use the substitution $u = \mathrm{s}(x)$ to show that

$$\int \frac{1}{(1-u^3)^{\frac{2}{3}}}\,\mathrm{d}u = \mathrm{s}^{-1}(u) + \text{constant}.$$

(v) Find, in terms of u, the integrals

$$\int \frac{1}{(1-u^3)^{\frac{4}{3}}}\,\mathrm{d}u \qquad \text{and} \qquad \int (1-u^3)^{\frac{1}{3}}\,\mathrm{d}u.$$