STEP I – Integration 1

Q1, (STEP I, 2004, Q2)

The square bracket notation $[x]$ means the greatest integer less than or equal to x. For example, $[\pi] = 3$, $[\sqrt{24}] = 4$ and $[5] = 5$.

(i) Sketch the graph of $y = \sqrt{[x]}$ and show that

$$\int_0^a \sqrt{[x]}\, \mathrm{d}x = \sum_{r=0}^{a-1} \sqrt{r}$$

when a is a positive integer.

(ii) Show that $\int_0^a 2^{[x]}\, \mathrm{d}x = 2^a - 1$ when a is a positive integer.

(iii) Determine an expression for $\int_0^a 2^{[x]}\, \mathrm{d}x$ when a is positive but not an integer.

Q2, (STEP I, 2004, Q4)

Differentiate $\sec t$ with respect to t.

(i) Use the substitution $x = \sec t$ to show that $\displaystyle\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}}\, \mathrm{d}x = \frac{\sqrt{3}-2}{8} + \frac{\pi}{24}$.

(ii) Determine $\displaystyle\int \frac{1}{(x+2)\sqrt{(x+1)(x+3)}}\, \mathrm{d}x$.

(iii) Determine $\displaystyle\int \frac{1}{(x+2)\sqrt{x^2+4x-5}}\, \mathrm{d}x$.

Q3, (STEP I, 2005, Q5)

(i) Evaluate the integral

$$\int_0^1 (x+1)^{k-1}\ \mathrm{d}x$$

in the cases $k \neq 0$ and $k = 0$.

Deduce that $\dfrac{2^k - 1}{k} \approx \ln 2$ when $k \approx 0$.

(ii) Evaluate the integral

$$\int_0^1 x\,(x+1)^m\ \mathrm{d}x$$

in the different cases that arise according to the value of m.

Q4, (STEP I, 2006, Q5)

(i) Use the substitution $u^2 = 2x + 1$ to show that, for $x > 4$,

$$\int \frac{3}{(x-4)\sqrt{2x+1}}\,\mathrm{d}x = \ln\left(\frac{\sqrt{2x+1}-3}{\sqrt{2x+1}+3}\right) + K\,,$$

where K is a constant.

(ii) Show that $\displaystyle\int_{\ln 3}^{\ln 8} \frac{2}{\mathrm{e}^x\sqrt{\mathrm{e}^x+1}}\,\mathrm{d}x = \frac{7}{12} + \ln\frac{2}{3}$.

Q5, (STEP I, 2006, Q3)

Prove the identities $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$ and $\cos^4\theta + \sin^4\theta \equiv 1 - \frac{1}{2}\sin^2 2\theta$. Hence or otherwise evaluate

$$\int_0^{\frac{\pi}{2}} \cos^4\theta\,\mathrm{d}\theta \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \sin^4\theta\,\mathrm{d}\theta\,.$$

Evaluate also

$$\int_0^{\frac{\pi}{2}} \cos^6\theta\,\mathrm{d}\theta \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \sin^6\theta\,\mathrm{d}\theta\,.$$

Q5, (STEP I, 2008, Q6)

The function f is defined by

$$\mathrm{f}(x) = \frac{\mathrm{e}^x - 1}{\mathrm{e} - 1}, \qquad x \geqslant 0,$$

and the function g is the inverse function to f, so that $\mathrm{g}(\mathrm{f}(x)) = x$. Sketch $\mathrm{f}(x)$ and $\mathrm{g}(x)$ on the same axes.

Verify, by evaluating each integral, that

$$\int_0^{\frac{1}{2}} \mathrm{f}(x)\,\mathrm{d}x + \int_0^{k} \mathrm{g}(x)\,\mathrm{d}x = \frac{1}{2(\sqrt{\mathrm{e}}+1)}\,,$$

where $k = \dfrac{1}{\sqrt{\mathrm{e}}+1}$, and explain this result by means of a diagram.

Q6, (STEP I, 2009, Q7)

Show that, for any integer m,

$$\int_0^{2\pi} \mathrm{e}^x \cos mx\,\mathrm{d}x = \frac{1}{m^2+1}(\mathrm{e}^{2\pi} - 1)\,.$$

(i) Expand $\cos(A+B) + \cos(A-B)$. Hence show that

$$\int_0^{2\pi} \mathrm{e}^x \cos x \cos 6x\,\mathrm{d}x = \tfrac{19}{650}(\mathrm{e}^{2\pi} - 1)\,.$$

(ii) Evaluate $\displaystyle\int_0^{2\pi} \mathrm{e}^x \sin 2x \sin 4x \cos x\,\mathrm{d}x\,.$

Q7, (STEP I, 2011, Q2)

The number E is defined by $E = \displaystyle\int_0^1 \frac{\mathrm{e}^x}{1+x}\,\mathrm{d}x$.

Show that

$$\int_0^1 \frac{x\mathrm{e}^x}{1+x}\,\mathrm{d}x = \mathrm{e} - 1 - E\,,$$

and evaluate $\displaystyle\int_0^1 \frac{x^2\mathrm{e}^x}{1+x}\,\mathrm{d}x$ in terms of e and E.

Evaluate also, in terms of E and e as appropriate:

(i) $\displaystyle\int_0^1 \frac{\mathrm{e}^{\frac{1-x}{1+x}}}{1+x}\,\mathrm{d}x$;

(ii) $\displaystyle\int_1^{\sqrt{2}} \frac{\mathrm{e}^{x^2}}{x}\,\mathrm{d}x$.

Q8, (STEP I, 2012, Q5)

Show that

$$\int_0^{\frac14\pi} \sin(2x)\ln(\cos x)\,\mathrm{d}x = \frac14(\ln 2 - 1)\,,$$

and that

$$\int_0^{\frac14\pi} \cos(2x)\ln(\cos x)\,\mathrm{d}x = \frac18(\pi - \ln 4 - 2)\,.$$

Hence evaluate

$$\int_{\frac14\pi}^{\frac12\pi} \big(\cos(2x) + \sin(2x)\big)\,\ln\big(\cos x + \sin x\big)\,\mathrm{d}x\,.$$
