STEP I – General Probability Questions (Sheet 2)

Probability

Understand and use mutually exclusive, independent, ***and complementary*** events when calculating probabilities. Link to discrete and continuous distributions.

Understand and use conditional probability, including the use of tree diagrams, Venn diagrams, two-way tables.

Know, understand and use the formula:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Know, understand and use the conditional probability formula:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Modelling with probability, including critiquing assumptions made and the likely effect of more realistic assumptions.

Q1, (STEP II, 2010, Q13)

Rosalind wants to join the Stepney Chess Club. In order to be accepted, she must play a challenge match consisting of several games against Pardeep (the Club champion) and Quentin (the Club secretary), in which she must win at least one game against each of Pardeep and Quentin. From past experience, she knows that the probability of her winning a single game against Pardeep is p and the probability of her winning a single game against Quentin is q, where $0 < p < q < 1$.

(i) The challenge match consists of three games. Before the match begins, Rosalind must choose either to play Pardeep twice and Quentin once or to play Quentin twice and Pardeep once. Show that she should choose to play Pardeep twice.

(ii) In order to ease the entry requirements, it is decided instead that the challenge match will consist of four games. Now, before the match begins, Rosalind must choose whether to play Pardeep three times and Quentin once (strategy 1), or to play Pardeep twice and Quentin twice (strategy 2) or to play Pardeep once and Quentin three times (strategy 3).

Show that, if $q - p > \frac{1}{2}$, Rosalind should choose strategy 1.

If $q - p < \frac{1}{2}$ give examples of values of p and q to show that strategy 2 can be better or worse than strategy 1.

Q2, (STEP II, 2011, Q12)

Xavier and Younis are playing a match. The match consists of a series of games and each game consists of three points.

Xavier has probability p and Younis has probability $1-p$ of winning the first point of any game. In the second and third points of each game, the player who won the previous point has probability p and the player who lost the previous point has probability $1-p$ of winning the point. If a player wins two consecutive points in a single game, the match ends and that player has won; otherwise the match continues with another game.

(i) Let w be the probability that Younis wins the match. Show that, for $p \neq 0$,

$$w = \frac{1-p^2}{2-p}.$$

Show that $w > \frac{1}{2}$ if $p < \frac{1}{2}$, and $w < \frac{1}{2}$ if $p > \frac{1}{2}$. Does w increase whenever p decreases?

(ii) If Xavier wins the match, Younis gives him £1; if Younis wins the match, Xavier gives him £k. Find the value of k for which the game is fair in the case when $p = \frac{2}{3}$.

(iii) What happens when $p = 0$?

Q3, (STEP I, 2013, Q12)

Each day, I have to take k different types of medicine, one tablet of each. The tablets are identical in appearance. When I go on holiday for n days, I put n tablets of each type in a container and on each day of the holiday I select k tablets at random from the container.

(i) In the case $k = 3$, show that the probability that I will select one tablet of each type on the first day of a three-day holiday is $\frac{9}{28}$.

Write down the probability that I will be left with one tablet of each type on the last day (irrespective of the tablets I select on the first day).

(ii) In the case $k = 3$, find the probability that I will select one tablet of each type on the first day of an n-day holiday.

(iii) In the case $k = 2$, find the probability that I will select one tablet of each type on each day of an n-day holiday, and use Stirling's approximation

$$n! \approx \sqrt{2n\pi}\left(\frac{n}{\mathrm{e}}\right)^n$$

to show that this probability is approximately $2^{-n}\sqrt{n\pi}$.

Q4, (STEP I , 2015, Q13)

A fair die with faces numbered 1, ..., 6 is thrown repeatedly. The events A, B, C, D and E are defined as follows.

A: the first 6 arises on the nth throw.
B: at least one 5 arises before the first 6.
C: at least one 4 arises before the first 6.
D: exactly one 5 arises before the first 6.
E: exactly one 4 arises before the first 6.

Evaluate the following probabilities:

(i) $\mathrm{P}(A)$ (ii) $\mathrm{P}(B)$ (iii) $\mathrm{P}(B \cap C)$ (iv) $\mathrm{P}(D)$ (v) $\mathrm{P}(D \cup E)$.

For some parts of this question, you may want to make use of the binomial expansion in the form:

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2}x^2 + \cdots + \frac{(n+r-1)!}{r!(n-1)!}x^r + \cdots .$$

Q5, (STEP II, 2015, Q12)

Four players A, B, C and D play a coin-tossing game with a fair coin. Each player chooses a sequence of heads and tails, as follows:

Player A: HHT; Player B: THH; Player C: TTH; Player D: HTT.

The coin is then tossed until one of these sequences occurs, in which case the corresponding player is the winner.

(i) Show that, if only A and B play, then A has a probability of $\frac{1}{4}$ of winning.

(ii) If all four players play together, find the probabilities of each one winning.

(iii) Only B and C play. What is the probability of C winning if the first two tosses are TT?

Let the probabilities of C winning if the first two tosses are HT, TH and HH be p, q and r, respectively. Show that $p = \frac{1}{2} + \frac{1}{2}q$.

Find the probability that C wins.

Q6, (2016, Q6)

Starting with the result $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, prove that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C).$$

Write down, without proof, the corresponding result for four events A, B, C and D.

A pack of n cards, numbered 1, 2, …, n, is shuffled and laid out in a row. The result of the shuffle is that each card is equally likely to be in any position in the row. Let E_i be the event that the card bearing the number i is in the ith position in the row. Write down the following probabilities:

(i) $P(E_i)$;

(ii) $P(E_i \cap E_j)$, where $i \neq j$;

(iii) $P(E_i \cap E_j \cap E_k)$, where $i \neq j$, $j \neq k$ and $k \neq i$.

Hence show that the probability that at least one card is in the same position as the number it bears is

$$1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n+1}\frac{1}{n!}.$$

Find the probability that exactly one card is in the same position as the number it bears.
