STEP I – General Probability Questions (Sheet 1)

Q1, (STEP I, 2004, Q14)

Three pirates are sharing out the contents of a treasure chest containing n gold coins and 2 lead coins. The first pirate takes out coins one at a time until he takes out one of the lead coins. The second pirate then takes out coins one at a time until she draws the second lead coin. The third pirate takes out all the gold coins remaining in the chest.

Find:

(i) the probability that the first pirate will have some gold coins;

(ii) the probability that the second pirate will have some gold coins;

(iii) the probability that all three pirates will have some gold coins.

Q2, (STEP II, 2004, Q14)

Explain why, if A, B and C are three events,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C),$$

where P(X) denotes the probability of event X.

A cook makes three plum puddings for Christmas. He stirs r silver sixpences thoroughly into the pudding mixture before dividing it into three equal portions. Find an expression for the probability that at least one pudding contains no sixpence. Show that the cook must stir 6 or more sixpences into the mixture if there is to be less than $\frac{1}{3}$ chance that at least one of the puddings contains no sixpence.

Given that the cook stirs 6 sixpences into the mixture and that each pudding contains at least one sixpence, find the probability that there are two sixpences in each pudding.

Q3, (STEP I, 2005, Q12)

(a) The probability that a hobbit smokes a pipe is 0.7 and the probability that a hobbit wears a hat is 0.4. The probability that a hobbit smokes a pipe but does not wear a hat is p. Determine the range of values of p consistent with this information.

(b) The probability that a wizard wears a hat is 0.7; the probability that a wizard wears a cloak is 0.8; and the probability that a wizard wears a ring is 0.4. The probability that a wizard does not wear a hat, does not wear a cloak and does not wear a ring is 0.05. The probability that a wizard wears a hat, a cloak and also a ring is 0.1. Determine the probability that a wizard wears exactly two of a hat, a cloak, and a ring.

The probability that a wizard wears a hat but not a ring, **given** that he wears a cloak, is q. Determine the range of values of q consistent with this information.

Q4, (STEP I, 2006, Q13)

A very generous shop-owner is hiding small diamonds in chocolate bars. Each diamond is hidden independently of any other diamond, and on average there is one diamond per kilogram of chocolate.

(i) I go to the shop and roll a fair six-sided die once. I decide that if I roll a score of N, I will buy $100N$ grams of chocolate. Show that the probability that I will have no diamonds is

$$\frac{e^{-0.1}}{6}\left(\frac{1-e^{-0.6}}{1-e^{-0.1}}\right)$$

Show also that the expected number of diamonds I find is 0.35.

(ii) Instead, I decide to roll a fair six-sided die repeatedly until I score a 6. If I roll my first 6 on my Tth throw, I will buy $100T$ grams of chocolate. Show that the probability that I will have no diamonds is

$$\frac{e^{-0.1}}{6-5e^{-0.1}}$$

Calculate also the expected number of diamonds that I find. (You may find it useful to consider the the binomial expansion of $(1-x)^{-2}$.)

Q5, (STEP I, 2006, Q14) **Q5, (STEP I, 2006, Q14)**

(i) A bag of sweets contains one red sweet and n blue sweets. I take a sweet from the bag, note its colour, return it to the bag, then shake the bag. I repeat this until the sweet I take is the red one. Find an expression for the probability that I take the red sweet on the rth attempt. What value of n maximises this probability?

(ii) Instead, I take sweets from the bag, without replacing them in the bag, until I take the red sweet. Find an expression for the probability that I take the red sweet on the rth attempt. What value of n maximises this probability?

Q6, (2007, Q13)

A bag contains eleven small discs, which are identical except that six of the discs are blank and five of the discs are numbered, using the numbers 1, 2, 3, 4 and 5. The bag is shaken, and four discs are taken one at a time without replacement.

Calculate the probability that:

(i) all four discs taken are numbered;

(ii) all four discs taken are numbered, given that the disc numbered "3" is taken first;

(iii) exactly two numbered discs are taken, given that the disc numbered "3" is taken first;

(iv) exactly two numbered discs are taken, given that the disc numbered "3" is taken;

(v) exactly two numbered discs are taken, given that a numbered disc is taken first;

(vi) exactly two numbered discs are taken, given that a numbered disc is taken.

Q7, (STEP II, 2008, Q13)

Bag P and bag Q each contain n counters, where $n \geqslant 2$. The counters are identical in shape and size, but coloured either black or white. First, k counters $(0 \leqslant k \leqslant n)$ are drawn at random from bag P and placed in bag Q. Then, k counters are drawn at random from bag Q and placed in bag P.

(i) If initially $n-1$ counters in bag P are white and one is black, and all n counters in bag Q are white, find the probability in terms of n and k that the black counter ends up in bag P.

Find the value or values of k for which this probability is maximised.

(ii) If initially $n-1$ counters in bag P are white and one is black, and $n-1$ counters in bag Q are white and one is black, find the probability in terms of n and k that the black counters end up in the same bag.

Find the value or values of k for which this probability is maximised.

Q8, (STEP I, 2009, Q12)

Prove that, for any real numbers x and y, $x^2 + y^2 \geqslant 2xy$.

(i) Carol has two bags of sweets. The first bag contains a red sweets and b blue sweets, whereas the second bag contains b red sweets and a blue sweets, where a and b are positive integers. Carol shakes the bags and picks one sweet from each bag without looking. Prove that the probability that the sweets are of the same colour cannot exceed the probability that they are of different colours.

(ii) Simon has three bags of sweets. The first bag contains a red sweets, b white sweets and c yellow sweets, where a, b and c are positive integers. The second bag contains b red sweets, c white sweets and a yellow sweets. The third bag contains c red sweets, a white sweets and b yellow sweets. Simon shakes the bags and picks one sweet from each bag without looking. Show that the probability that exactly two of the sweets are of the same colour is

$$\frac{3(a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2)}{(a+b+c)^3},$$

and find the probability that the sweets are all of the same colour. Deduce that the probability that exactly two of the sweets are of the same colour is at least 6 times the probability that the sweets are all of the same colour.
