STEP I – Differentiation 2

Q1, (STEP I, 2014, Q4)

An accurate clock has an hour hand of length a and a minute hand of length b (where $b > a$), both measured from the pivot at the centre of the clock face. Let x be the distance between the ends of the hands when the angle between the hands is θ, where $0 \leqslant \theta < \pi$.

Show that the rate of increase of x is greatest when $x = (b^2 - a^2)^{\frac{1}{2}}$.

In the case when $b = 2a$ and the clock starts at mid-day (with both hands pointing vertically upwards), show that this occurs for the first time a little less than 11 minutes later.

Q2, (STEP I, 2015, Q1)

(i) Sketch the curve $y = \mathrm{e}^x(2x^2 - 5x + 2)$.

Hence determine how many real values of x satisfy the equation $\mathrm{e}^x(2x^2 - 5x + 2) = k$ in the different cases that arise according to the value of k.

You may assume that $x^n \mathrm{e}^x \to 0$ as $x \to -\infty$ for any integer n.

(ii) Sketch the curve $y = \mathrm{e}^{x^2}(2x^4 - 5x^2 + 2)$.

Q3, (STEP I, 2016, Q2)

Differentiate, with respect to x,

$$(ax^2 + bx + c)\ln\left(x + \sqrt{1 + x^2}\right) + (dx + e)\sqrt{1 + x^2},$$

where a, b, c, d and e are constants. You should simplify your answer as far as possible.

Hence integrate:

(i) $\ln\left(x + \sqrt{1 + x^2}\right)$;

(ii) $\sqrt{1 + x^2}$;

(iii) $x\ln\left(x + \sqrt{1 + x^2}\right)$.

Q4, (STEP I, 2016, Q4)

(i) Differentiate $\dfrac{z}{(1 + z^2)^{\frac{1}{2}}}$ with respect to z.

(ii) The *signed curvature* κ of the curve $y = \mathrm{f}(x)$ is defined by

$$\kappa = \frac{\mathrm{f}''(x)}{\left(1 + (\mathrm{f}'(x))^2\right)^{\frac{3}{2}}}.$$

Use this definition to determine all curves for which the signed curvature is a non-zero constant. For these curves, what is the geometrical significance of κ?

Q5, (STEP I, 2017, Q3)

The points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$, where $p > 0$ and $q < 0$, lie on the curve C with equation

$$y^2 = 4ax\,,$$

where $a > 0$. Show that the equation of the tangent to C at P is

$$y = \frac{1}{p}x + ap\,.$$

The tangents to the curve at P and at Q meet at R. These tangents meet the y-axis at S and T respectively, and O is the origin. Prove that the area of triangle OPQ is twice the area of triangle RST.

Q6, (STEP I, 2017, Q5)

A circle of radius a is centred at the origin O. A rectangle $PQRS$ lies in the minor sector OMN of this circle where M is $(a, 0)$ and N is $(a\cos\beta, a\sin\beta)$, and β is a constant with $0 < \beta < \frac{\pi}{2}$. Vertex P lies on the positive x-axis at $(x, 0)$; vertex Q lies on ON; vertex R lies on the arc of the circle between M and N; and vertex S lies on the positive x-axis at $(s, 0)$.

Show that the area A of the rectangle can be written in the form

$$A = x(s - x)\tan\beta\,.$$

Obtain an expression for s in terms of a, x and β, and use it to show that

$$\frac{\mathrm{d}A}{\mathrm{d}x} = (s - 2x)\tan\beta - \frac{x^2}{s}\tan^3\beta\,.$$

Deduce that the greatest possible area of rectangle $PQRS$ occurs when $s = x(1 + \sec\beta)$ and show that this greatest area is $\frac{1}{2}a^2\tan\frac{1}{2}\beta$.

Show also that this greatest area occurs when $\angle ROS = \frac{1}{2}\beta$.

Q7, (STEP I, 2018, Q1)

The line $y = a^2x$ and the curve $y = x(b - x)^2$, where $0 < a < b$, intersect at the origin O and at points P and Q. The x-coordinate of P is less than the x-coordinate of Q. Find the coordinates of P and Q, and sketch the line and the curve on the same axes.

Show that the equation of the tangent to the curve at P is

$$y = a(3a - 2b)x + 2a(b - a)^2.$$

This tangent meets the y-axis at R. The area of the region between the curve and the line segment OP is denoted by S. Show that

$$S = \frac{1}{12}(b - a)^3(3a + b)\,.$$

The area of triangle OPR is denoted by T. Show that $S > \frac{1}{3}T$.
