Q1, (STEP I, 2005, Q2)

The point P has coordinates $(p^2, 2p)$ and the point Q has coordinates $(q^2, 2q)$, where p and q are non-zero and $p \neq q$. The curve C is given by $y^2 = 4x$. The point R is the intersection of the tangent to C at P and the tangent to C at Q. Show that R has coordinates $(pq, p+q)$.

The point S is the intersection of the normal to C at P and the normal to C at Q. If p and q are such that $(1, 0)$ lies on the line PQ, show that S has coordinates $(p^2 + q^2 + 1,\ p+q)$, and that the quadrilateral $PSQR$ is a rectangle.

Q2, (STEP I, 2009, Q2)

A curve has the equation

$$y^3 = x^3 + a^3 + b^3 ,$$

where a and b are positive constants. Show that the tangent to the curve at the point $(-a, b)$ is

$$b^2 y - a^2 x = a^3 + b^3 .$$

In the case $a = 1$ and $b = 2$, show that the x-coordinates of the points where the tangent meets the curve satisfy

$$7x^3 - 3x^2 - 27x - 17 = 0 .$$

Hence find positive integers p, q, r and s such that

$$p^3 = q^3 + r^3 + s^3 .$$

Q3, (STEP I, 2010, Q2)

The curve $y = \left(\dfrac{x-a}{x-b}\right)\mathrm{e}^x$, where a and b are constants, has two stationary points. Show that

$$a - b < 0 \quad \text{or} \quad a - b > 4 .$$

(i) Show that, in the case $a = 0$ and $b = \frac{1}{2}$, there is one stationary point on either side of the curve's vertical asymptote, and sketch the curve.

(ii) Sketch the curve in the case $a = \frac{9}{2}$ and $b = 0$.

Q4, (STEP I, 2011, Q4)

The distinct points P and Q, with coordinates $(ap^2, 2ap)$ and $(aq^2, 2aq)$ respectively, lie on the curve $y^2 = 4ax$. The tangents to the curve at P and Q meet at the point T. Show that T has coordinates $(apq, a(p+q))$. You may assume that $p \neq 0$ and $q \neq 0$.

The point F has coordinates $(a, 0)$ and ϕ is the angle TFP. Show that

$$\cos\phi = \frac{pq+1}{\sqrt{(p^2+1)(q^2+1)}}$$

and deduce that the line FT bisects the angle PFQ.

Q5, (STEP I, 2012, Q1)

The line L has equation $y = c - mx$, with $m > 0$ and $c > 0$. It passes through the point $R(a, b)$ and cuts the axes at the points $P(p, 0)$ and $Q(0, q)$, where a, b, p and q are all positive. Find p and q in terms of a, b and m.

As L varies with R remaining fixed, show that the minimum value of the sum of the distances of P and Q from the origin is $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$, and find in a similar form the minimum distance between P and Q. (You may assume that any stationary values of these distances are minima.)

Q6, (STEP I, 2012, Q2)

(i) Sketch the curve $y = x^4 - 6x^2 + 9$ giving the coordinates of the stationary points.

Let n be the number of distinct real values of x for which

$$x^4 - 6x^2 + b = 0.$$

State the values of b, if any, for which (a) $n = 0$; (b) $n = 1$; (c) $n = 2$; (d) $n = 3$; (e) $n = 4$.

(ii) For which values of a does the curve $y = x^4 - 6x^2 + ax + b$ have a point at which both $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$?

For these values of a, find the number of distinct real values of x for which

$$x^4 - 6x^2 + ax + b = 0,$$

in the different cases that arise according to the value of b.

(iii) Sketch the curve $y = x^4 - 6x^2 + ax$ in the case $a > 8$.

Q7, (STEP I, 2012, Q4)

The curve C has equation $xy = \frac{1}{2}$. The tangents to C at the distinct points $P\left(p, \frac{1}{2p}\right)$ and $Q\left(q, \frac{1}{2q}\right)$, where p and q are positive, intersect at T and the normals to C at these points intersect at N. Show that T is the point

$$\left(\frac{2pq}{p+q}, \frac{1}{p+q}\right).$$

In the case $pq = \frac{1}{2}$, find the coordinates of N. Show (in this case) that T and N lie on the line $y = x$ and are such that the product of their distances from the origin is constant.