STEP I – Binomial Distribution

Statistical distributions

Understand informally the concept of a random variable.

Understand and use simple, discrete probability distributions (calculation of expectation and variance of discrete random variables is ***included***), including the Binomial distribution as a model; calculate probabilities using the Binomial distribution.

Q1, (STEP I, 2004, Q12)

In a certain factory, microchips are made by two machines. Machine A makes a proportion λ of the chips, where $0 < \lambda < 1$, and machine B makes the rest. A proportion p of the chips made by machine A are perfect, and a proportion q of those made by machine B are perfect, where $0 < p < 1$ and $0 < q < 1$. The chips are sorted into two groups: group 1 contains those that are perfect and group 2 contains those that are imperfect.

In a large random sample taken from group 1, it is found that $\frac{2}{5}$ were made by machine A. Show that λ can estimated as

$$\frac{2q}{3p+2q} \,.$$

Subsequently, it is discovered that the sorting process is faulty: there is a probability of $\frac{1}{4}$ that a perfect chip is assigned to group 2 and a probability of $\frac{1}{4}$ that an imperfect chip is assigned to group 1. Taking into account this additional information, obtain a new estimate of λ.

Q2, (STEP II, 2006, Q2)

A cricket team has only three bowlers, Arthur, Betty and Cuba, each of whom bowls 30 balls in any match. Past performance reveals that, on average, Arthur takes one wicket for every 36 balls bowled, Betty takes one wicket for every 25 balls bowled, and Cuba takes one wicket for every 41 balls bowled.

(i) In one match, the team took exactly one wicket, but the name of the bowler was not recorded. Using a binomial model, find the probability that Arthur was the bowler.

(ii) Show that the average number of wickets taken by the team in a match is approximately 3. Give with brief justification a suitable model for the number of wickets taken by the team in a match and show that the probability of the team taking at least five wickets in a given match is approximately $\frac{1}{5}$.

[You may use the approximation $e^3 = 20$.]

Q3, (STEP I, 2016, Q12)

(i) Alice tosses a fair coin twice and Bob tosses a fair coin three times. Calculate the probability that Bob gets more heads than Alice.

(ii) Alice tosses a fair coin three times and Bob tosses a fair coin four times. Calculate the probability that Bob gets more heads than Alice.

(iii) Let p_1 be the probability that Bob gets the same number of heads as Alice, and let p_2 be the probability that Bob gets more heads than Alice, when Alice and Bob each toss a fair coin n times.

Alice tosses a fair coin n times and Bob tosses a fair coin $n+1$ times. Express the probability that Bob gets more heads than Alice in terms of p_1 and p_2, and hence obtain a generalisation of the results of parts (i) and (ii).

Q4, (STEP II, 2012, Q12)

A modern villa has complicated lighting controls. In order for the light in the swimming pool to be on, a particular switch in the hallway must be on and a particular switch in the kitchen must be on. There are four identical switches in the hallway and four identical switches in the kitchen. Guests cannot tell whether the switches are on or off, or what they control.

Each Monday morning a guest arrives, and the switches in the hallway are either all on or all off. The probability that they are all on is p and the probability that they are all off is $1-p$. The switches in the kitchen are each on or off, independently, with probability $\frac{1}{2}$.

(i) On the first Monday, a guest presses one switch in the hallway at random and one switch in the kitchen at random. Find the probability that the swimming pool light is on at the end of this process. Show that the probability that the guest has pressed the swimming pool light switch in the hallway, given that the light is on at the end of the process, is $\dfrac{1-p}{1+2p}$.

(ii) On each of seven Mondays, guests go through the above process independently of each other, and each time the swimming pool light is found to be on at the end of the process. Given that the most likely number of days on which the swimming pool light switch in the hallway was pressed is 3, show that $\frac{1}{4} < p < \frac{5}{14}$.

Q5, (STEP II, 2016, Q13)

(i) The random variable X has a binomial distribution with parameters n and p, where $n = 16$ and $p = \frac{1}{2}$. Show, using an approximation in terms of the standard normal density function $\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}x^2}$, that

$$\mathrm{P}(X = 8) \approx \frac{1}{2\sqrt{2\pi}}\,.$$

(ii) By considering a binomial distribution with parameters $2n$ and $\frac{1}{2}$, show that

$$(2n)! \approx \frac{2^{2n}(n!)^2}{\sqrt{n\pi}}\,.$$

(iii) By considering a Poisson distribution with parameter n, show that

$$n! \approx \sqrt{2\pi n}\,\mathrm{e}^{-n}\,n^n\,.$$