STEP I – Algebra and Functions 2

Q1, (STEP I, 2010, Q1)

Given that

$$5x^2 + 2y^2 - 6xy + 4x - 4y \equiv a\,(x - y + 2)^2 + b\,(cx + y)^2 + d\,,$$

find the values of the constants a, b, c and d.

Solve the simultaneous equations

$$5x^2 + 2y^2 - 6xy + 4x - 4y = 9\,,$$
$$6x^2 + 3y^2 - 8xy + 8x - 8y = 14\,.$$

Q2, (STEP I, 2011, Q8)

(i) The numbers m and n satisfy

$$m^3 = n^3 + n^2 + 1\,. \qquad (*)$$

(a) Show that $m > n$. Show also that $m < n + 1$ if and only if $2n^2 + 3n > 0$. Deduce that $n < m < n + 1$ unless $-\frac{3}{2} \leqslant n \leqslant 0$.

(b) Hence show that the only solutions of $(*)$ for which both m and n are integers are $(m, n) = (1, 0)$ and $(m, n) = (1, -1)$.

(ii) Find all integer solutions of the equation

$$p^3 = q^3 + 2q^2 - 1\,.$$

Q3, (STEP I, 2015, Q7)

Let

$$\mathrm{f}(x) = 3ax^2 - 6x^3$$

and, for each real number a, let $\mathrm{M}(a)$ be the greatest value of $\mathrm{f}(x)$ in the interval $-\frac{1}{3} \leqslant x \leqslant 1$. Determine $\mathrm{M}(a)$ for $a \geqslant 0$. [The formula for $\mathrm{M}(a)$ is different in different ranges of a; you will need to identify three ranges.]

Q4, (STEP I, 2018, Q2)

If $x = \log_b c$, express c in terms of b and x and prove that $\dfrac{\log_a c}{\log_a b} = \log_b c\,.$

(i) Given that $\pi^2 < 10$, prove that

$$\frac{1}{\log_2 \pi} + \frac{1}{\log_5 \pi} > 2\,.$$

(ii) Given that $\log_2 \dfrac{\pi}{\mathrm{e}} > \dfrac{1}{5}$ and that $\mathrm{e}^2 < 8$, prove that $\ln \pi > \frac{17}{15}$.

(iii) Given that $\mathrm{e}^3 > 20$, $\pi^2 < 10$ and $\log_{10} 2 > \frac{3}{10}$, prove that $\ln \pi < \frac{15}{13}$.

Q5, (STEP I, 2017, Q2)

(i) The inequality $\frac{1}{t} \leqslant 1$ holds for $t \geqslant 1$. By integrating both sides of this inequality over the interval $1 \leqslant t \leqslant x$, show that

$$\ln x \leqslant x - 1 \qquad (*)$$

for $x \geqslant 1$. Show similarly that $(*)$ also holds for $0 < x \leqslant 1$.

(ii) Starting from the inequality $\frac{1}{t^2} \leqslant \frac{1}{t}$ for $t \geqslant 1$, show that

$$\ln x \geqslant 1 - \frac{1}{x} \qquad (**)$$

for $x > 0$.

(iii) Show, by integrating $(*)$ and $(**)$, that

$$\frac{2}{y+1} \leqslant \frac{\ln y}{y-1} \leqslant \frac{y+1}{2y}$$

for $y > 0$ and $y \neq 1$.

Q6, (STEP I, 2018, Q5)

(i) Write down the most general polynomial of degree 4 that leaves a remainder of 1 when divided by any of $x-1$, $x-2$, $x-3$ or $x-4$.

(ii) The polynomial $\mathrm{P}(x)$ has degree N, where $N \geqslant 1$, and satisfies

$$\mathrm{P}(1) = \mathrm{P}(2) = \cdots = \mathrm{P}(N) = 1\,.$$

Show that $\mathrm{P}(N+1) \neq 1$.

Given that $\mathrm{P}(N+1) = 2$, find $\mathrm{P}(N+r)$ where r is a positive integer. Find a positive integer r, independent of N, such that $\mathrm{P}(N+r) = N+r$.

(iii) The polynomial $\mathrm{S}(x)$ has degree 4. It has integer coefficients and the coefficient of x^4 is 1. It satisfies

$$\mathrm{S}(a) = \mathrm{S}(b) = \mathrm{S}(c) = \mathrm{S}(d) = 2001\,,$$

where a, b, c and d are distinct (not necessarily positive) integers.

(a) Show that there is no integer e such that $\mathrm{S}(e) = 2018$.

(b) Find the number of ways the (distinct) integers a, b, c and d can be chosen such that $\mathrm{S}(0) = 2017$ and $a < b < c < d$.

Q7, (STEP I, 2013, Q8)

(i) The functions a, b, c and d are defined by

$$\begin{aligned} \mathrm{a}(x) &= x^2 \quad (-\infty < x < \infty), \\ \mathrm{b}(x) &= \ln x \quad (x > 0), \\ \mathrm{c}(x) &= 2x \quad (-\infty < x < \infty), \\ \mathrm{d}(x) &= \sqrt{x} \quad (x \geqslant 0)\,. \end{aligned}$$

Write down the following composite functions, giving the domain and range of each:

cb, ab, da, ad.

(ii) The functions f and g are defined by

$$\begin{aligned} \mathrm{f}(x) &= \sqrt{x^2 - 1} \quad (|x| \geqslant 1), \\ \mathrm{g}(x) &= \sqrt{x^2 + 1} \quad (-\infty < x < \infty). \end{aligned}$$

Determine the composite functions fg and gf, giving the domain and range of each.

(iii) Sketch the graphs of the functions h and k defined by

$$\begin{aligned} \mathrm{h}(x) &= x + \sqrt{x^2 - 1} \quad (x \geqslant 1), \\ \mathrm{k}(x) &= x - \sqrt{x^2 - 1} \quad (|x| \geqslant 1), \end{aligned}$$

justifying the main features of the graphs, and giving the equations of any asymptotes. Determine the domain and range of the composite function kh.

Q8, (STEP I, 2018, Q7)

(i) In the cubic equation $x^3 - 3pqx + pq(p+q) = 0$, where p and q are distinct real numbers, use the substitution

$$x = \frac{pz + q}{z + 1}$$

to show that the equation reduces to $az^3 + b = 0$, where a and b are to be expressed in terms of p and q.

(ii) Show further that the equation $x^3 - 3cx + d = 0$, where c and d are non-zero real numbers, can be written in the form $x^3 - 3pqx + pq(p+q) = 0$, where p and q are distinct real numbers, provided $d^2 > 4c^3$.

(iii) Find the real root of the cubic equation $x^3 + 6x - 2 = 0$.

(iv) Find the roots of the equation $x^3 - 3p^2x + 2p^3 = 0$, and hence show how the equation $x^3 - 3cx + d = 0$ can be solved in the case $d^2 = 4c^3$.
