STEP I – Algebra and Functions 1

Q1 (STEP I, 2004, Q3)

(i) Show that $x - 3$ is a factor of

$$x^3 - 5x^2 + 2x^2y + xy^2 - 8xy - 3y^2 + 6x + 6y\,. \qquad (*)$$

Express $(*)$ in the form $(x-3)(x+ay+b)(x+cy+d)$ where a, b, c and d are integers to be determined.

(ii) Factorise $6y^3 - y^2 - 21y + 2x^2 + 12x - 4xy + x^2y - 5xy^2 + 10$ into three linear factors.

Q2, (STEP I, 2005, Q3)

In this question a and b are distinct, non-zero real numbers, and c is a real number.

(i) Show that, if a and b are either both positive or both negative, then the equation

$$\frac{x}{x-a} + \frac{x}{x-b} = 1$$

has two distinct real solutions.

(ii) Show that the equation

$$\frac{x}{x-a} + \frac{x}{x-b} = 1 + c$$

has exactly one real solution if $c^2 = -\dfrac{4ab}{(a-b)^2}$. Show that this condition can be written $c^2 = 1 - \left(\dfrac{a+b}{a-b}\right)^2$ and deduce that it can only hold if $0 < c^2 \leqslant 1$.

Q3, (STEP I, 2006, Q3)

In this question b, c, p and q are real numbers.

(i) By considering the graph $y = x^2 + bx + c$ show that $c < 0$ is a sufficient condition for the equation $x^2 + bx + c = 0$ to have distinct real roots. Determine whether $c < 0$ is a necessary condition for the equation to have distinct real roots.

(ii) Determine necessary and sufficient conditions for the equation $x^2 + bx + c = 0$ to have distinct positive real roots.

(iii) What can be deduced about the number and the nature of the roots of the equation $x^3 + px + q = 0$ if $p > 0$ and $q < 0$?

What can be deduced if $p < 0$ and $q < 0$? You should consider the different cases that arise according to the value of $4p^3 + 27q^2$.

Q4, (STEP I, 2006, Q2)

A small goat is tethered by a rope to a point at ground level on a side of a square barn which stands in a large horizontal field of grass. The sides of the barn are of length $2a$ and the rope is of length $4a$. Let A be the area of the grass that the goat can graze. Prove that $A \leqslant 14\pi a^2$ and determine the minimum value of A.

Q5 (STEP I, 2007, Q4)

Show that $x^3 - 3xbc + b^3 + c^3$ can be written in the form $(x + b + c)\,\text{Q}(x)$, where $\text{Q}(x)$ is a quadratic expression. Show that $2\text{Q}(x)$ can be written as the sum of three expressions, each of which is a perfect square.

It is given that the equations $ay^2 + by + c = 0$ and $by^2 + cy + a = 0$ have a common root k. The coefficients a, b and c are real, a and b are both non-zero, and $ac \neq b^2$. Show that

$$\left(ac - b^2\right) k = bc - a^2$$

and determine a similar expression involving k^2. Hence show that

$$\left(ac - b^2\right)\left(ab - c^2\right) = \left(bc - a^2\right)^2$$

and that $a^3 - 3abc + b^3 + c^3 = 0$. Deduce that either $k = 1$ or the two equations are identical.

Q6, (STEP I, 2007, Q6)

(i) Given that $x^2 - y^2 = (x - y)^3$ and that $x - y = d$ (where $d \neq 0$), express each of x and y in terms of d. Hence find a pair of integers m and n satisfying $m - n = \left(\sqrt{m} - \sqrt{n}\right)^3$ where $m > n > 100$.

(ii) Given that $x^3 - y^3 = (x - y)^4$ and that $x - y = d$ (where $d \neq 0$), show that $3xy = d^3 - d^2$. Hence show that

$$2x = d \pm d\sqrt{\frac{4d-1}{3}}$$

and determine a pair of distinct positive integers m and n such that $m^3 - n^3 = (m - n)^4$.

Q7, (STEP I, 2009, Q3)

(i) By considering the equation $x^2 + x - a = 0$, show that the equation $x = (a - x)^{\frac{1}{2}}$ has one real solution when $a \geqslant 0$ and no real solutions when $a < 0$.

Find the number of distinct real solutions of the equation

$$x = \left((1 + a)x - a\right)^{\frac{1}{3}}$$

in the cases that arise according to the value of a.

(ii) Find the number of distinct real solutions of the equation

$$x = (b + x)^{\frac{1}{2}}$$

in the cases that arise according to the value of b.

Q8, (STEP I, 2008, Q3)

Prove that, if $c \geqslant a$ and $d \geqslant b$, then

$$ab + cd \geqslant bc + ad\,. \qquad (*)$$

(i) If $x \geqslant y$, use $(*)$ to show that $x^2 + y^2 \geqslant 2xy$.

If, further, $x \geqslant z$ and $y \geqslant z$, use $(*)$ to show that $z^2 + xy \geqslant xz + yz$ and deduce that $x^2 + y^2 + z^2 \geqslant xy + yz + zx$.

Prove that the inequality $x^2 + y^2 + z^2 \geqslant xy + yz + zx$ holds for all x, y and z.

(ii) Show similarly that the inequality

$$\frac{s}{t} + \frac{t}{r} + \frac{r}{s} \geqslant 3$$

holds for all positive r, s and t.

[**Note:** The final part of this question differs (though not substantially) from what appeared in the actual examination since this was found to be unsatisfactory (though not incorrect) in a way that had not been anticipated.]