Integration: Which Method To Use

1. Is it of the form $f(ax+b)$? \(\Rightarrow\) Reverse Chain Rule

\[
\int \frac{2}{5-3x} \, dx = \int (5-3x)^{-1} \, dx = \frac{2\ln|5-3x|}{-3} + c
\]

\[\Rightarrow \text{by diff of bracket}\]

\[= -\frac{2}{3} \ln|5-3x| + c\]

2. Is it of the form $\frac{f'(x)}{f(x)}$?

\[
\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + c
\]

\[
\int f'(x) (f(x))^n \, dx = \frac{1}{n+1} (f(x))^{n+1} + c
\]

3. Check for partial fractions (i.e. is the bottom factored or will it?)

\[\int \frac{1}{x^2 - 2} \, dx\]

\[= \int \frac{1}{(x - \sqrt{2})(x + \sqrt{2})} \, dx\]

Let \(\frac{1}{(x - \sqrt{2})(x + \sqrt{2})} = \frac{A}{x + \sqrt{2}} + \frac{B}{x - \sqrt{2}}\)

\[\Rightarrow 1 = A(x - \sqrt{2}) + B(x + \sqrt{2})\]

Let \(x = \sqrt{2} \Rightarrow 1 = B(2\sqrt{2}) \Rightarrow B = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}\]
\[ x = -\sqrt{2} \Rightarrow 1 = A(-2\sqrt{2}) = \frac{-\sqrt{2}}{4} \]

\[ \Rightarrow \int \frac{-\sqrt{2}}{4} \left( \frac{1}{x + \sqrt{2}} + \frac{1}{x - \sqrt{2}} \right) \, dx \]

\[ = \frac{\sqrt{2}}{4} \left( \ln \left| \frac{x + \sqrt{2}}{x - \sqrt{2}} \right| + c \right) \]

(4) Check if trig identities are applicable

\[ \sin^2 x + \cos^2 x = 1 \]

\[ \cos 2x = 2\cos^2 x - 1 \]

\[ 1 + \cot^2 x = \csc^2 x \]

\[ \tan^2 x + 1 = \sec^2 x \]

\[ \sin 2x = 2\sin x \cos x \]

**Differentiation**

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( f'(x) )</th>
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<tbody>
<tr>
<td>( \tan kx )</td>
<td>( k \sec^2 kx )</td>
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<tr>
<td>( \sec x )</td>
<td>( \sec x \tan x )</td>
</tr>
<tr>
<td>( \cot x )</td>
<td>( -\csc^2 x )</td>
</tr>
<tr>
<td>( \cosec x )</td>
<td>( -\cosec x \cot x )</td>
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\[ \int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx = \tan x + c \]

\[ \int \sec^6 x \tan x \, dx = \int \frac{\sec^5 x \sec x \tan x \, dx}{\sec x} \]

\[ \int f'(x)(f(x))^n \, dx = \frac{1}{n+1}(f(x))^{n+1} + c \]

\[ = \frac{1}{6} \sec^6 x + c \]

\[ \int \cot^2 x \, dx = \int \cosec^2 x - 1 \, dx = -\cot x - x + c \]

\[ \sin^2 x + \cos^2 x = 1 \]

\[ 1 + \cot^2 x = \cosec^2 x \]
\[ \int \frac{4 \sin x \cos x}{2 \cos^2 x - 3} \, dx = \int \frac{2(2 \sin x \cos x)}{(2 \cos^2 x - 1) - 2} \, dx \]
\[ = \int \frac{2 \sin 2x}{\cos 2x - 2} \, dx = -\int \frac{-2 \sin 2x}{\cos 2x - 2} \, dx \]
\[ = -\ln |\cos 2x - 2| + c \]

\[ \int (\tan x + 1)^2 \, dx = \int \tan^2 x + 2 \tan x + 1 \, dx \]

\[ \int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx = \tan x - x + c \]

\[ \int 2 \tan x \, dx = \int \frac{2 \sin x}{\cos x} \, dx = -2 \int -\sin x \, dx \]
\[ = -2 \ln |\cos x| + c \]

\[ \therefore I = \tan x - x - 2 \ln |\cos x| + c \]
\[ = \tan x - 2 \ln |\cos x| + c \]

\[ \int (6 + 3 \sin 6x)^2 \, dx = \int 36 + 36 \sin 6x + 9 \sin^2 6x \, dx \]

\[ \int 9 \sin^2 (6x) = \int \frac{9}{2} - \frac{9}{2} \cos (12x) \, dx \]
\[ = \frac{9}{2} x - \frac{9}{24} \sin (12x) + c \]
\[ = \frac{9}{2} x - \frac{3}{8} \sin (12x) + c \]

\[ \frac{\cos^2 \Theta}{2} = 1 - 2 \sin^2 (\Theta) \]
\[ \cos 2 \alpha = 1 - 2 \sin^2 (6x) \]
\[ = 2 \sin^2 (6x) = 1 - \cos (12x) \]

\[ \sin^2 (6x) = \frac{1}{2} - \frac{1}{2} \cos (12x) \]
Substitution or By Parts

- By parts is usually used to integrate a function that is a product of two functions that are not related by differentiation.

  e.g. \( \int xe^x \, dx \checkmark \)

  \( \int x^2 \sin x \, dx \)

  \( \int x^2e^{x^3+4} \, dx \times \) (Differential of power is outside: use reverse chain rule or substitution)

- Substitution is used for examples like the third one above where there is a differential relationship
  
  \( (i.e. \frac{d}{dx}(x^2+4) = 3x^2 \Rightarrow x^2 \text{ at front will cancel}) \)

  or where several terms on a denominator need to be combined to make the integral simpler.

  e.g. \( \int x\sqrt{6x - 3} \, dx \) (By substitution)

Let \( u = 6x - 3 \Rightarrow \frac{du}{dx} = 6 \Rightarrow dx = \frac{1}{6} \, du \)

\[ \Rightarrow I = \int xu^{\frac{1}{2}} \, du = \frac{1}{6} \int xu^{\frac{1}{2}} \, du \]

Since \( u = 6x - 3 \Rightarrow u + 3 = 6x \Rightarrow x = \frac{u + 3}{6} \)

\[ \therefore I = \frac{1}{6} \int \left( \frac{u + 3}{6} \right) u^{\frac{1}{2}} \, du = \frac{1}{6} \int \frac{u^{\frac{3}{2}}}{6} + \frac{u^{\frac{1}{2}}}{2} \, du \]

\[ = \frac{1}{6} \left[ \frac{2}{3} u^{\frac{3}{2}} + \frac{2}{3} u^{\frac{1}{2}} \right] + c = \frac{1}{90} u^{\frac{3}{2}} + \frac{1}{18} u^{\frac{1}{2}} + c \]

\[ = \frac{1}{60} (6x - 3)^{\frac{3}{2}} + \frac{1}{18} (6x - 3)^{\frac{1}{2}} + c \]
e.g. \[ \int x \sqrt{6x - 3} \, dx \]

Integration by parts \[ \int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx \]

\[ u = x \Rightarrow \frac{du}{dx} = 1 \]
\[ v = \frac{2}{3} (6x - 3)^{\frac{3}{2}} \Rightarrow \frac{dv}{dx} = (6x - 3)^{\frac{1}{2}} \]
\[ = \frac{1}{9} (6x - 3)^{\frac{3}{2}} \]

\[ \therefore I = x \times \frac{1}{9} (6x - 3)^{\frac{3}{2}} - \int \frac{1}{9} (6x - 3)^{\frac{3}{2}} \, dx \]
\[ = \frac{x}{9} (6x - 3)^{\frac{3}{2}} - \frac{1}{9} \times \frac{2}{5} (6x - 3)^{\frac{5}{2}} + c \]
\[ = \frac{x}{9} (6x - 3)^{\frac{3}{2}} - \frac{1}{135} (6x - 3)^{\frac{5}{2}} + c \]

However it looks like we have got two different answers to the same question.

For substitution we got
\[ \frac{x}{90} (6x - 3)^{\frac{3}{2}} + \frac{1}{18} (6x - 3)^{\frac{5}{2}} + c \]
\[ = \frac{1}{90} (6x - 3)^{\frac{5}{2}} [15x - (6x - 3)] + c \]
\[ = \frac{2}{90} (6x - 3)^{\frac{5}{2}} (3x + 1) + c \]
\[ = \frac{1}{45} (6x - 3)^{\frac{5}{2}} (3x + 1) + c \]

For by parts we got
\[ \frac{x}{90} (6x - 3)^{\frac{3}{2}} - \frac{1}{135} (6x - 3)^{\frac{5}{2}} + c \]
\[ = \frac{1}{135} (6x - 3)^{\frac{5}{2}} [15x - (6x - 3)] + c \]
\[ = \frac{3}{135} (6x - 3)^{\frac{5}{2}} (3x + 1) + c \]
\[ = \frac{1}{45} (6x - 3)^{\frac{5}{2}} (3x + 1) + c \]

\[ \therefore \text{Both answers equivalent} \]