Geometric Sequence

A geometric sequence is one that has a common ratio or multiple between terms.

\[ \text{e.g. } 2, 4, 8, 16, 32, \ldots \quad (a = 2, \, r = 2) \]

\[ 100, 50, 25, 12.5, \ldots \quad (a = 100, \, r = \frac{1}{2}) \]

[Note: we can obtain the common ratio between terms by dividing a term by its previous term.]

\[ n^{th} \text{ Term of a Geometric Sequence} \]

If \( a \) denotes the first term and \( r \) the common ratio between terms, then:

\[
\begin{align*}
    u_1 &= a \\
    u_2 &= ar \\
    u_3 &= ar^2 \\
    \vdots \\
    u_n &= ar^{n-1}
\end{align*}
\]

For a geometric sequence

\[ u_n = ar^{n-1} \]

(Must remember this)

Sum of \( n \) Terms of a Geometric Sequence

\[
S_n = \frac{a(1 - r^n)}{1 - r} \quad \text{or} \quad \frac{a(r^n - 1)}{r - 1} \quad \text{(Given in formulae)}
\]

Proof

\[
S_n = a + ar + ar^2 + \ldots + ar^{n-1}
\]

\[
rS_n = \quad ar + ar^2 + \ldots + ar^{n-1} + ar^n
\]

\[
S_n - rS_n = a + 0 + 0 + \ldots + 0 - ar^n
\]
\[ S_n = \frac{a(1-r^n)}{1-r} \]

For each of the following geometric progressions, find an expression for the \( n \)th term.

a) 1, 5, 25, 125, …

\[ a = 1, \quad r = 5 \]

\[ u_n = 1 \times 5^{n-1} = 5^{n-1} \]

b) 3, –12, 48, –192, …

\[ a = 3, \quad r = -4 \]

\[ u_n = 3 \times (-4)^{n-1} \]

\[ \sum_{n=1}^{6} g^{n+1} = 64 + 32 \]  

\[ a = 64 \]  

\[ r = 8 \]  

\[ S_6 = \frac{64(1-8^6)}{1-8} \]

\[ = 2396736 \]

The second and fifth terms of a geometric series are 0.5 and 32 respectively.

a) Find the first term and common ratio of the series.

\[ ar = 0.5 \]

\[ ar^4 = 32 \]

\[ \frac{ar^4}{ar} = \frac{32}{0.5} \]

\[ \Rightarrow r^3 = 64 \]  

\[ \Rightarrow r = \sqrt[3]{64} = 4 \]

\[ \Rightarrow a \times 4 = 0.5 \]  

\[ \Rightarrow a = \frac{1}{8} \]

b) Find the number of terms of the series that are smaller than 10,000.

\[ ar^{n-1} < 10,000 \]  

\[ \Rightarrow \frac{1}{8}(4)^{n-1} < 10,000 \]
\[
4^n < 80,000 \quad \Rightarrow \quad \log_4(4^n) < \log_4(80,000)
\]
\[
\Rightarrow \quad n - 1 < \log_4(80,000)
\]
\[
\Rightarrow \quad n < \log_4(80,000) + 1
\]
\[
\Rightarrow \quad n < 9.14
\]
\[
\Rightarrow \quad n = 9 \quad (n \text{ must be an integer})
\]