Concavity, Convexity, and Point of Inflection:

An object is convex if its surface protrudes outward.

E.g.

\[ y = \sin x \]

An object is concave if its surface protrudes inward.

E.g.

The concept is similar for functions. Take, for example:

From \( 0 < x < \pi \), the curve is concave.

From \( \pi < x < 2\pi \), the curve is convex.

Technical definition:

- A curve is concave in an interval if for all values in that interval \( \frac{d^2y}{dx^2} < 0 \).
- A curve is convex if \( \frac{d^2y}{dx^2} > 0 \) in an interval.
Points of Inflection

A point of inflection is a point at which the function changes from being concave to concave or concave to convex. At this point, the curve is neither concave nor convex.

Fact: Point of inflection $\Rightarrow f''(x) = 0$

Finding the points where $\frac{d^2y}{dx^2} = 0$

For $y = x^3$, $\frac{dy}{dx} = 3x^2$, $\frac{d^2y}{dx^2} = 6x$

$\Rightarrow 6x = 0 \Rightarrow x = 0, y = 0$

$\therefore (0, 0)$

For $y = x^4$, $\frac{dy}{dx} = 4x^3$, $\frac{d^2y}{dx^2} = 12x^2$

$\Rightarrow 12x^2 = 0 \Rightarrow x = 0, y = 0$

$\therefore (0, 0)$
To test for a point of inflection we must find the gradient slightly to the left and slightly to the right of our suspected point of inflection.

For \( y = x^3 \)

\[
\begin{align*}
  f'(0.1) &= 3(0.1)^2 = 0.03 > 0 \\
  f'(0) &= 3(0)^2 = 0 \\
  f'(0.1) &= 3(0.1)^2 = 0.03 > 0
\end{align*}
\]

\[\text{Stationary point of inflection} \quad \text{i.e. both}\]

For \( y = x^4 \)

\[
\begin{align*}
  f'(0.1) &= 4(-0.1)^3 = -0.004 < 0 \\
  f'(0) &= 4(0)^3 = 0 \\
  f'(0.1) &= 4(0.1)^3 = 0.004 > 0
\end{align*}
\]

\[\text{Stationary point only}\]

**Example**

For each of the following functions, find the interval on which the function is:

i) convex   
ii) concave

a) \( f(x) = x^3 - 3x^2 + x - 2 \)

\[
\begin{align*}
  f'(x) &= 3x^2 - 6x + 1 \\
  f''(x) &= 6x - 6
\end{align*}
\]

i) For concave, \( 6x - 6 > 0 \) \( \Rightarrow \) \( 6x < 6 \) \( \Rightarrow x < 1 \)

\[\text{convex when } x < 1\]

ii) For concave, \( 6x - 6 < 0 \) \( \Rightarrow x > 1 \)

\[\text{concave for } x > 1\]