Arithmetic Sequence

An arithmetic sequence is a list of numbers in which neighboring terms have a common difference.

e.g. $1, 3, 5, 7, \ldots$
$6, 4, 2, 0, \ldots$
$1, 2, 4, 8, 16, \ldots$

The first term of an arithmetic sequence is denoted by $a_1$, the common difference by $d$ and the term number by $n$. The $n^{th}$ term is denoted by $a_n$.

e.g. for $5, 9, 13, 17, \ldots$
$$a_1 = 5, d = 4, a_3 = 13, \text{ etc.}$$

Arithmetic Series

The word ‘series’ describes the act of adding all of the terms in a sequence together.

e.g. The sequence $2, 5, 8, \ldots$ has series $2 + 5 + 8 + \ldots$

Sigma Notation

$$\sum_{n=1}^{4} 2n + 1$$

This means evaluate $2n + 1$ for $n = 1$ to $n = 4$, $n \in \mathbb{N}$ and sum the answers.

These are called limits

$$2(1)+1 + 2(2)+1 + 2(3)+1 + 2(4)+1 = 24$$
e.g. Find \( \sum_{n=1}^{4} 2 \)

Refer to the variable that we are incrementing. If the variable does not appear in the expression we still evaluate the expression.

\[
\begin{align*}
2 + 2 + 2 + 2 &= 8 \\
\text{at} & \quad \text{at} & \quad \text{at} & \quad \text{at} \\
n=1 & \quad n=2 & \quad n=3 & \quad n=4
\end{align*}
\]

**Formulæ for Arithmetic Series**

\( n^{th} \) term: \( u_1 = a \)

\[
u_2 = a + d \\
u_3 = a + 2d \\
\vdots \\
u_n = a + (n-1)d
\]

**Summing Series**

\( S_n \) denotes summing the first \( n \) terms of a series.

\( \text{e.g.} \) for the sequence 1, 7, 13, 19, 25

\( S_3 \) is the sum of the first 3 terms \( \Rightarrow S_3 = 1 + 7 + 13 = 21 \)

\[
S_n = \frac{1}{2} n [2a + (n-1)d]
\]

where \( n \) is the position of the last term

\( a \) is the first term

\( d \) is the common difference

**Proof:**

\[
S_n = a + (a + d) + (a + 2d) + (a + 3d) + \ldots + (a + (n-1)d) = \]

\[
= na + d + 2d + 3d + \ldots + (n-1)d
\]
\[ S_n \text{ in reverse is } S_n = (n-1)d + (n-2)d + \ldots + 3d + 2d + d + na \]

Adding \( S_n \) to its reversed version:

\[
S_n + S_n = 2S_n = \begin{cases} \frac{na}{2} & d \end{cases} + \begin{cases} \frac{na}{2} & d \\ (n-1)d & (n-2)d & (n-3)d & \ldots & d \end{cases}
\]

\[ \Rightarrow 2S_n = 2na + nd + nd + nd + \ldots + nd \]

\[ \Rightarrow 2S_n = 2na + (n-1)nd \]

\[ \Rightarrow S_n = \frac{1}{2}n\left[2a + (n-1)d\right] \]

e.g. Find \[ \sum_{n=1}^{100} 2n + 3 \]

1. Write out the first few terms of the sequence to find \( a \) and \( d \)

\[ a_1 = 2(1) + 3 = 5 \quad a_2 = 2(2) + 3 = 7 \quad a_3 = 2(3) + 3 = 9 \]

\[ a = 5 \]
\[ d = 2 \]

\[ n = 100 \quad \text{No of terms we are summing} \]

2. Use \( S_n \) formula

\[ \Rightarrow S_{100} = \frac{1}{2}(100)\left[2(5) + 99(2)\right] = 10,400 \]
The third and eighth terms of an arithmetic series are 72 and 37 respectively.

\[ a + 2d = 72 \]
\[ a + 7d = 37 \]

\[ \begin{array}{c}
\{ \\
1x + 1x + 7y = 72 \\
1x + 7y = 37
\} \end{array} \]

\[ x = 8 \text{ (Make use of calculator if full working not required)} \]
\[ y = -7 \]

b/ \[ S_{25} = \frac{1}{2}(25)[2(86) + 24(-7)] = 50 \]

Summing series in which the first term is not \( n = 1 \)

e.g. \[ \sum_{n=21}^{59} 6n + 1 \]

Method 1: \[ S_{59} - S_{20} \]
\[ a = 7 \]
\[ d = 6 \]
\[ n = 59 \text{ or } 20 \]
\[ S_{59} - S_{20} = \frac{1}{2}(59)[2(7) + 58(6)] - \frac{1}{2}(20)[2(7) + 19(6)] \]
\[ = 10,679 - 1280 = 9,399 \]

Method 2: \[ S_n = \frac{1}{2}n[a + l] \]

\[ a \text{ is when } n = 21 \]
\[ l \text{ is when } n = 59 \]
\[ a + (n-1)d \]
\[ a = 6(21) + 1 = 127 \]
\[ l = 6(59) + 1 = 355 \]
\[ n = 59 - 21 + 1 = 39 \]
\[ S = \frac{1}{2}(39)[127 + 355] = 9,399 \]

To count 1st term

This method is more efficient in algebraic rather than numerical situations i.e. when \( a, d \) or \( n \) is unknown.
A sequence of terms \( u_1, u_2, u_3, \ldots \) is defined by
\[
    u_n = 2n + 5, \quad \text{for } n \geq 1.
\]

(i) Write down the values of \( u_1, u_2 \) and \( u_3 \). \[2\]

(ii) State what type of sequence it is. \[1\]

(iii) Given that \( \sum_{n=1}^{N} u_n = 2200 \), find the value of \( N \). \[5\]

\( u_1 = 7 \), \( u_2 = 9 \), \( u_3 = 11 \)

\( \text{Arithmetic} \)

\[ S_N = \frac{1}{2} N \left[ 2a + (N-1)d \right] \]

\( a = 7 \) \( \quad d = 2 \)

\[
    \therefore \quad \frac{1}{2} N \left[ 2(7) + (N-1)2 \right] = 2200
\]

\[
    \Rightarrow \quad \frac{1}{2} N \left[ 14 + 2N - 2 \right] = 2200
\]

\[
    \Rightarrow \quad N \left[ 12 + 2N \right] = 4400
\]

\[
    \Rightarrow 12N + 2N^2 = 4400
\]

\[
    \Rightarrow 2N^2 + 12N - 4400 = 0
\]

\[
    \Rightarrow N = 44 \quad \text{or} \quad -50 \quad N \in \mathbb{N}
\]
A sequence \(u_1, u_2, u_3, \ldots\) is defined by

\[ u_1 = 8 \quad \text{and} \quad u_{n+1} = u_n + 3. \]

\[ \text{NEXT TERM} = \text{PREV TERM} + 3 \]  

(i) Show that \(u_5 = 20\).

(ii) The \(n\)th term of the sequence can be written in the form \(u_n = pn + q\). State the values of \(p\) and \(q\).

(iii) State what type of sequence it is.

(iv) Find the value of \(N\) such that \(\sum_{n=1}^{2N} u_n - \sum_{n=1}^N u_n = 1256\).

\[ u_1 = 8, \quad u_2 = 8 + 3 = 11, \quad u_3 = 11 + 3 = 14, \quad u_4 = 14 + 3 = 17, \quad u_5 = 17 + 3 = 20 \]

\[ a = 8, \quad d = 3 \]

\[ u_n = a + (n-1)d = 8 + (n-1)3 \]

\[ = 8 + 3n - 3 = 3n + 5 \]

(iii) \(\text{Arithmetic} \)

\[ S_{2N} - S_N = \frac{1}{2}(2N)[2(8) + (2N-1)3] - \frac{1}{2}(N)[2(8) + (N-1)3] \]

\[ \Rightarrow N[16 + 6N - 3] - \frac{1}{2}N[16 + 3N - 3] = 1256 \]

\[ \Rightarrow N[13 + 6N] - \frac{1}{2}N[13 + 3N] = 1256 \]

\[ \Rightarrow 2N[13 + 6N] - N[13 + 3N] = 2512 \]

\[ \Rightarrow 26N + 12N^2 - 13N - 3N^2 = 2512 \]

\[ \Rightarrow 9N^2 + 13N - 2512 = 0 \]

\[ \Rightarrow N = 16 \quad \text{or} \quad -\frac{157}{9}, \quad N \in \mathbb{N} \]