Mode

The mode of a CHV is the maximum of its probability density function (if a maximum exists).

The continuous random variable $X$ has probability density function given by

$$f(x) = \begin{cases} \frac{3}{80} (8 + 2x - x^2) & 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

a. Sketch the probability density function of $X$.

b. Find the mode of $X$.

$$a, \quad f(x) = -\frac{3}{80} (x^2 - 2x - 8) = \frac{3}{80} (x - 4)(x + 2)$$

roots at $x = 4$ or $x = -2$

$$F(0) = -\frac{3}{80} (8) = \frac{3}{10} \quad \text{LHS in } (0, \frac{3}{10})$$

$$F(4) = -\frac{3}{80} (4^2 - 2(4) - 8) = 0 \quad \text{RHS in } (4, 0)$$

b. The mode may or may not be a local maximum as found by differentiating.

in this P.D.F differentiating and finding the stationary point would lead us to a mode outside of the domain of validity of the P.D.F.

Here, from observing where the function is valid, we see the mode is $F(0)$.

[In this question, the mode is found by differentiation]
\[ f(x) = \begin{cases} \frac{3}{80}(8 + 2x - x^2) & 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases} \]

\[ f'(x) = \frac{3}{80}(2 - 2x) = 0 \quad (\text{i.e., mode on this function is the root point}) \]

\[ \Rightarrow 2 - 2x = 0 \quad \Rightarrow x = 1 \]

Mode is x-value paired with highest "frequency".

**Median**

The median is the value of \( x \) to the left of which 0.5 of the probability lies and to the right of which 0.5 of the probability lies.

**Method 1: Using P.D.F.**

Solve \( \int_x^\infty f(x) \, dx = 0.5 \) or \( \int_{-\infty}^x f(x) \, dx = 0.5 \)

**Method 2: Use C.D.F.**

Solve \( F(x) = 0.5 \)
The continuous random variable \( X \) has cumulative distribution function given by

\[
F(x) = \begin{cases} 
0 & \quad x < 0 \\
\frac{x^2}{6} & \quad 0 \leq x < 2 \\
\frac{-x^2}{3} + 2x - 2 & \quad 2 \leq x \leq 3 \\
1 & \quad x > 3
\end{cases}
\]

Find the following, giving your answers to 3 decimal places:

a) the median value of \( X \)

b) the quartiles and the interquartile range of \( X \).

\( a \) \( F(x) = 0.5 \implies \frac{x^2}{6} = 0.5 \implies x^2 = 3 \implies x = \sqrt{3} \)

\((-\sqrt{3} \text{ not valid since } -\sqrt{3} < 0) \implies x = \sqrt{3} \) (within \( 0 \leq x < 2 \) \implies valid)

\( b \) \( F(x) = 0.75 \) (Upper quartile)

\[
\implies \frac{x^2}{6} = 0.75 \implies x^2 = 4.5 \implies x = \pm \frac{3\sqrt{2}}{2}
\]

\( 0 \neq \pm \frac{3\sqrt{2}}{2} \neq 2 \) \implies disregard.

\( F(x) = 0.75 \implies -\frac{x^2}{3} + 2x - 2 = 0.75 \)

\[
\implies -\frac{x^2}{3} + 2x - 2.75 = 0 \implies x = \frac{6 \pm \sqrt{5}}{2}, \quad \frac{6 - \sqrt{5}}{2}
\]

\( \therefore \text{UQ} = \frac{6 - \sqrt{5}}{2} \approx 2.136 \)

\( \text{LQ} : F(x) = 0.25 \implies \frac{x^2}{6} = 0.25 \implies x^2 = \frac{3}{2} \implies x = \frac{\sqrt{6}}{2} \) or \(-\frac{\sqrt{6}}{2} \)

\( \implies \text{LQ} = \frac{\sqrt{6}}{2} \approx 1.225 \)

\[ \text{IQR} = \frac{6 - \sqrt{5}}{2} - \frac{\sqrt{6}}{2} \approx 0.909 \]
Skewness of Probability Density Function

Positive skew refers to a function that looks to have been stretched in the positive direction.

Positive skew $\Rightarrow$ Mode < Median or Mean

Negative skew refers to a function that looks to have been stretched in the negative direction.

Negative skew $\Rightarrow$ Mode > Median or Mean