Volumes of Revolution

This is the volume covered when an area is rotated around an axis:

\[ y = x^2 \]

When this area is rotated 360° around the x-axis, a solid shape is formed.

The shape formed looks like this:

**Proposition**

The formula for working out the volume of such a solid formed by rotating a curve around the x-axis is:

\[ V = \pi \int_{a}^{b} y^2 \, dx \]

**Proof (Sort of)**

Consider the volume generated to be approximated by a number of discs.
\[ V \approx \pi y_1^2 w + \pi y_2^2 w + \ldots + \pi y_n^2 w \]
\[ = \pi \sum_{i=1}^{n} y_i^2 w \]

We can make this a better approximation by using more and thinner discs, i.e., by making \( w \) smaller.

Letting \( w \to 0 \),

we get
\[ \sum_{i=1}^{n} y_i^2 w \to \int_{a}^{b} y^2 \, dx \]
\[ \therefore V \to \pi \int_{a}^{b} y^2 \, dx \]

The above becomes an integral as \( w \to 0 \),

the discs become infinitely thin and integration is the summation of the infinitely thin.

e.g. Find volume of the solid formed when the following area is rotated 360° around the \( x \)-axis.

\[ y = x^3 \]

Using \( V = \pi \int_{a}^{b} y^2 \, dx \)

\[ y = x^3 \to y^2 = x^6 \]
\[ \therefore V = \pi \int_{a}^{b} x^6 \, dx = \pi \left[ \frac{1}{7} x^7 \right]_{a}^{b} \]
\[
\pi \left[ \frac{1}{4} (4)^2 - \frac{1}{4} (2)^2 \right] = \frac{16 \pi}{7}
\]

**Must not forget \( \pi \)**

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e.g. Find the volume of the solid formed when \( y = \frac{1}{\sqrt{x-3}} \) is rotated 360°, considering the area between \( x = 5 \), \( x = 6 \) and the \( x \)-axis.

\[
y = \frac{1}{\sqrt{x-3}} \quad \Rightarrow \quad y^2 = \frac{1}{x-3}
\]

\[
.: \quad V = \pi \int_5^6 \frac{1}{x-3} \, dx = \pi \left[ \ln |x-3| \right]_5^6
\]

\[
= \pi \left( \ln 3 - \ln 2 \right) = 1.274 \quad \text{(4sf)}
\]

**BAD!**

**GOOD**

Keep answers exact when irrational!!!
The diagram shows the curve with equation \( y = \frac{1}{\sqrt{3x + 2}} \). The shaded region is bounded by the curve and the lines \( x = 0 \), \( x = 2 \) and \( y = 0 \).

(i) Find the exact area of the shaded region.

\[
\int_{0}^{2} \left(3x + 2\right)^{\frac{1}{2}} \, dx = \left[ \frac{1}{3} \left(3x + 2\right)^{\frac{3}{2}} \right]_{0}^{2} = \left[ \frac{2}{3} (3x + 2)^{\frac{3}{2}} \right]_{0}^{2}
\]

\[
= \frac{2}{3} (3(2) + 2)^{\frac{3}{2}} - \frac{2}{3} (3(0) + 2)^{\frac{3}{2}}
\]

\[
= \frac{2}{3} \sqrt{8} - \frac{2}{3} \sqrt{2}
\]

\[
= \frac{4}{3} \sqrt{2} - \frac{2}{3} \sqrt{2} = \frac{2}{3} \sqrt{2}
\]

(ii) The shaded region is rotated completely about the \( x \)-axis. Find the exact volume of the solid formed, simplifying your answer.

\[
V = \pi \int_{0}^{2} \frac{1}{3x+2} \, dx = \pi \left[ \frac{1}{3} \ln |3x + 2| \right]_{0}^{2}
\]

\[
= \pi \left[ \frac{1}{3} \ln 8 - \frac{1}{3} \ln 2 \right] = \frac{1}{3} \pi (\ln 8 - \ln 2) = \frac{1}{3} \pi \ln \left( \frac{8}{2} \right) = \frac{1}{3} \pi \ln 4
\]
Volume Generated When Rotating Around the y-Axis

e.g. Find the volume generated when the following area is rotated around the y-axis 360°.

![Graph showing a curve and the formula for volume calculation.]

Formula: \( V = \pi \int_{a}^{b} x^2 \, dy \)

1. Rearrange to say “\( x = f(y) \)”.

\[
y = x^2 \Rightarrow x = \sqrt{y} \quad (\text{Not } \frac{dy}{dx} \text{ since } x > 0 \text{ in diagram})
\]

2. Use above volume formula being sure to use y-limits, not x-limits.

\[\begin{align*}
x & = 3 \Rightarrow y = 3^2 = 9 \\
x & = 5 \Rightarrow y = 5^2 = 25 \\
x^2 & = y
\end{align*}\]

\[
V = \pi \int_{9}^{25} x^2 \, dy = \pi \int_{9}^{25} y \, dy = \pi \left[ \frac{y^3}{3} \right]_{9}^{25}
\]

\[
= \pi \left[ \frac{25^3}{3} - \frac{9^3}{3} \right] = \pi \times 2722 = 2722\pi
\]