Tangents and Normals to Circles

A tangent is a line that just touches a circle but does not cut it.

A normal (or radius) is at right angles to a tangent.

e.g. Find the equation of the tangent to the circle $x^2 + y^2 + 4x = 13$ at $(-1, 4)$

\[
(x+2)^2 - 4 + y^2 = 13 \
\Rightarrow (x+2)^2 + y^2 = 17
\]

\[\therefore \text{C}(-2, 0)\]

\[
\text{Gradient of normal:} \quad m = \frac{4 - 0}{-1 - (-2)} = 4
\]

\[\therefore \text{Gradient of tangent} = \frac{-1}{4} \\quad \text{Nag, reciprocal of normal gradient} \]

\[\therefore y - 4 = \frac{-1}{4}(x - (-1)) \]

\[\therefore 4y - 16 = -(2x + 1) \Rightarrow 4y - 16 = -2x - 1 \]

\[\Rightarrow x + 4y - 15 = 0\]
The line with equation \( y = x + k \) is a tangent to the circle with equation \( x^2 + y^2 + 6x - 8y + 17 = 0 \). Find the two possible values of \( k \).

Solving simultaneously:

\[
\begin{align*}
   x^2 + (x+k)^2 + 6x - 8(x+k) + 17 &= 0 \\
   \Rightarrow x^2 + x^2 + 2kx + k^2 + 6x - 8x - 8k + 17 &= 0 \\
   \Rightarrow 2x^2 + 2kx - 2x + k^2 - 8k + 17 &= 0
\end{align*}
\]

\( x^2 \) terms (underlined), \( x \) terms (underlined), constant (underlined)

\( \Rightarrow 2x^2 + x(2k-2) + k^2 - 8k + 17 = 0 \)

Take out \( x \) as a factor

\[
\begin{align*}
   a &= 2 \\
   b &= 2k-2 \\
   c &= k^2 - 8k + 17
\end{align*}
\]

\[
\begin{align*}
   b^2 - 4ac &= 0 \\
   (2k-2)^2 - 4(2)(k^2 - 8k + 17) &= 0 \\
   \Rightarrow 4k^2 - 8k + 4 - 8(k^2 - 8k + 17) &= 0 \\
   \Rightarrow 4k^2 - 8k + 4 - 8k^2 + 64k - 136 &= 0 \\
   \Rightarrow -4k^2 + 56k - 132 &= 0
\end{align*}
\]

\( k = 11 \text{ or } k = 3 \)