Proof

Notation

\( \Rightarrow \) means "implies that" or "leads to the direct consequence"

e.g. \[ 2x = 6 \Rightarrow x = 3 \]

\[ x^2 = 9 \not\Rightarrow x = 3 \]

Means "does not imply". This is because in this case \( x \) could be \(-3\).

\[ x^2 = 9 \Leftrightarrow x = 3 \]

This statement is true because \( x^2 = 9 \) is a direct consequence of \( x = 3 \).

\[ 2x = 6 \Leftrightarrow x = 3 \]

A "double way implication" is also referred to as "if and only if" or "iff".

Q1, (Jun 2006, Q4)
In each of the following cases choose one of the statements

\[ P \Rightarrow Q \]
\[ P \Leftrightarrow Q \]
\[ P \Leftrightarrow Q \]

to describe the complete relationship between \( P \) and \( Q \).

(i) \( P: \quad x^2 + x - 2 = 0 \)

\( Q: \quad x = 1 \)

\[ (x+2)(x-1) = 0 \Rightarrow x = -2, x = 1 \]

\[ P \Leftrightarrow Q \]

[1]

(ii) \( P: \quad y^3 > 1 \)

\( Q: \quad y > 1 \)

\[ P \Leftrightarrow Q \]

[1]
**Proof by Deduction**

This is proof in which we start with a statement we know to be true and deduce other true statements from that.

**Proof using Odd and Even Properties of Numbers**

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**Definitions**

- An even number is any number that can be written in the form $2n$ where $n$ is an integer.
- An odd number is any number that can be written in the form $2m+1$ or $2m-1$ where $m$ is an integer.

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e.g. Prove that the product of two odd numbers is also odd.

1. **Define your starting numbers**
   
   Let $2m+1$ and $2n+1$ be the two odd numbers in question.

2. **Perform whatever operation was requested**
   
   $$(2m+1)(2n+1) = 4mn + 2n + 2m + 1$$

3. **Factorise in a way that proves evenness or oddness**
   
   $$2(2mn + n + m) + 1$$  
   Exaggerate by leaving $+1$ outside the bracket

4. **Conclude by spelling out what you have just shown**

   Since this is of the form $2A+1$ we can deduce the product of the odd numbers is odd.
Proof by Exhaustion

In this type of proof you consider every possible case to validate the proof.

E.g. Prove by exhaustion that 47 is prime.

The idea here is that we will check every possible factor of 47 to see if an integer is obtained.

\[
\frac{47}{2} = 23.5 \quad : \text{not a factor}
\]

\[
\frac{47}{3} = 15.666... \quad : \text{not a factor}
\]

\[
\frac{47}{4} = 11.75 \quad : \text{not a factor}
\]

\[
\frac{47}{5} = 9.4 \quad : \text{not a factor}
\]

\[
\frac{47}{6} = 7.833... \quad : \text{not a factor}
\]

\[
\therefore \text{Since all integers } 2 < n < \sqrt{47} \text{ are not factors of } 47,
\]

\[
\Rightarrow 47 \text{ is prime}
\]
An alternative method using the calculator is given as follows:

1. Go to table mode

2. Input the number we are checking is prime divided by a variable $x$

   \[ f(x) = \frac{47}{x} \]

3. Set start, end and increment values

   \[ \text{start } x = 2 \]
   \[ \text{end integer closest to but less than } \sqrt{47} \]

4. Read off answer

   \[ \text{Proof by Counterexample} \]

   “It takes one bullet to kill a man” Vladimir Vladimirov (2007)

   i.e. one counterexample collapses an argument

   Disprove the following statement by means of a counterexample:

   \[ 2^n - 1 \text{ is always prime for any positive integer } n \geq 2 \]

   \[ n = 4 \implies 2^n - 1 = 16 - 1 = 15 \]
Set Notation

When dealing with proofs, it is often the case that we use shorthand notation to abbreviate sets of numbers.

- E means “is a member of”
- N means “the set of natural numbers” which is all integers \( \geq 1 \), i.e. \( N = \{1, 2, 3, 4, \ldots \} \)
- R means “the set of real numbers”
- Z means “the set of all integers” i.e. \( Z = \{\ldots -2, -1, 0, 1, 2, \ldots \} \)
- Q means “the set of all rational numbers” i.e. \( Q = \left\{ \frac{a}{b} : a, b \in Z \right\} \) such that \( a \) and \( b \) are integers
- C means “the set of complex numbers” i.e. \( C = \{a + bi : a, b \in R\} \)