Straight lines

Gradient of a line segment

Find the gradient of this line segment.

\[ m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \]

where \((x_1, y_1) \) and \((x_2, y_2)\) are two points.

\[ m = \frac{10 - 2}{6 - 1} = \frac{8}{5} \]

or

\[ m = \frac{2 - 10}{1 - 6} = \frac{-8}{-5} = \frac{8}{5} \]

Midpoint of a line segment

The midpoint of a line segment is given by the mean \(x\) and \(y\) coordinates of its endpoints.

\[ \text{Mid Point} = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \]

Length of a line segment

\[ \text{Length} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]
General Equation of a Straight Line

When written in the form \( y = mx + c \), we can easily read off the gradient \( m \) and the \( y \)-intercept \( c \).

\[ 6x + 5y = 3 \]

\[ 5y = 3 - 6x \]

\[ y = \frac{3}{5} - \frac{6x}{5} \]

\[ y = c + mx \quad \text{gradient} = -\frac{6}{5} \quad \text{y-intercept} = \frac{3}{5} \]

However, there are other forms which a straight line equation can take.

General Equation of Straight Line

\[ y - y_1 = m(x - x_1) \]

where \((x_1, y_1)\) is any point on the line

e.g. Find the equation of the line that passes through \((2, 7)\) and has gradient \(4\), in the form \(y = mx + c\).

\[ m = 4 \quad (x_1, y_1) = (2, 7) \]

\[ y - 7 = 4(x - 2) \]

\[ y - 7 = 4x - 8 \quad \Rightarrow \quad y = 4x - 1 \]
e.g. Find the equation of the line that passes through 
\((14, 7), (9, -12)\) in the form \(ax + by + c = 0\) where \(a, b, c \in \mathbb{Z}\)

\[m = \frac{-12 - 7}{9 - 14} = \frac{-19}{-5} = \frac{19}{5}\]

\((x_1, y_1) = (14, 7)\) — Could use \((9, -12)\) but easier to keep positive numbers

\[y - 7 = \frac{19}{5}(x - 14)\]

\[5y - 35 = 19(x - 14)\]

\[5y - 35 = 19x - 266\]

\[-19x + 5y + 231 = 0\]

which is in the required form \(ax + by + c = 0\)

At this point if we multiply both sides by \(5\), we get only integers

\[5y - 35 = 95x - 1330\]

\[-19x + 5y + 231 = 0\]

Any integer multiple of this answer is acceptable

\[e.g. -19000x + 5000y + 231000 = 0\]

is acceptable, but idiotic

Q2 (Jan 2012, Q8)

The line \(l\) has gradient \(-2\) and passes through the point \(A(3, 5)\). \(B\) is a point on the line \(l\) such that the distance \(AB\) is \(6\sqrt{5}\). Find the coordinates of each of the possible points \(B\).

\[E_{\text{eq}} \text{ of } AB: y - 5 = -2(x - 3) \Rightarrow y - 5 = -2x + 6\]

\[\Rightarrow y = -2x + 11\]

Using Pythagoras:

\[
\sqrt{(-2x + 6)^2 + (x - 3)^2} = 6\sqrt{5}
\]
\[(6 - 2x)^2 + (x - 3)^2 = 180\]
\[\Rightarrow 36 - 24x + 4x^2 + x^2 - 6x + 9 = 180\]
\[\Rightarrow 5x^2 - 30x - 135 = 0\]
\[\Rightarrow x^2 - 6x - 27 = 0\]
\[\Rightarrow (x - 9)(x + 3) = 0\]
\[\Rightarrow x = 9 \quad x = -3\]

If \(x = 9\), \(y = -2(9) + 11 = -7\) \(\therefore B(9, -7)\)

If \(x = -3\), \(y = -2(-3) + 11 = 17\) \(\therefore B(-3, 17)\)