A linear transformation of a distribution is performing a transformation to a variable of the form \( y = mX + c \).

e.g. Take the variable \( X \) with the distribution

\[
\begin{array}{c|c|c|c}
\text{x} & 1 & 2 & 3 \\
\hline
\text{p(x=x)} & \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\
\end{array}
\]

\[
E(X) = 1 \times \frac{1}{3} + 2 \times \frac{1}{2} + 3 \times \frac{1}{6} = \frac{11}{6} = E(X)
\]

\[
E(X^2) = 1^2 \times \frac{1}{3} + 2^2 \times \frac{1}{2} + 3^2 \times \frac{1}{6} = \frac{23}{6} = E(X^2)
\]

\[
\Rightarrow \text{Var}(X) = \frac{23}{6} - \left(\frac{11}{6}\right)^2 = \frac{17}{36} = \text{Var}(X)
\]

Now take the distribution of \( 3X \)

\[
\begin{array}{c|c|c|c}
\text{x} & 3 & 6 & 9 \\
\hline
\text{p(x=x)} & \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\
\end{array}
\]

\[
E(3X) = 3 \times \frac{1}{3} + 6 \times \frac{1}{2} + 9 \times \frac{1}{6} = \frac{11}{2} = 3E(X)
\]

\[
E((3X)^2) = 3^2 \times \frac{1}{3} + 6^2 \times \frac{1}{2} + 9^2 \times \frac{1}{6} = \frac{69}{2}
\]

\[
\text{Var}(3X) = \frac{69}{2} - \left(\frac{11}{2}\right)^2 = \frac{17}{4} = 9 \times \text{Var}(X)
\]

Reason for relationship:

Consider the points

\[
\begin{array}{cccc}
\text{0} & \text{1} & \text{2} & \text{3} & \text{4} \\
\text{Mean/middle point} \\
\end{array}
\]

If we \( x \) by \( 3 \)

\[
\begin{array}{cccc}
\text{0} & \text{3} & \text{6} & \text{i2} \\
\end{array}
\]

Everything is \( 3 \times \) bigger and \( 3 \times \) more spread out.

\[
\text{the middle is } 3 \times \text{ bigger } \Rightarrow E(3X) = 3E(X)
\]

Since the spread is bigger and the variance increases.
squares of spreads, the new variance will be multiplied by 9.

\[ E(kX) = kE(X) \]
\[ \text{Var}(kX) = k^2 \text{Var}(X) \]

Consider the distribution of \( X+3 \)

\[ \begin{array}{c|ccc}
   x & 4 & 5 & 6 \\
   p(x=x) & \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\
\end{array} \]

\[ E(X+3) = \frac{29}{6} = E(X) + 3 \]
\[ E((X+3)^2) = \frac{143}{6} \]

\[ \text{Var}(X+3) = \frac{17}{36} = \text{Var}(X) \]

Moving all points the same distance moves the centre of the data set but does nothing to the spread.

\[ E(X+c) = E(X) + c \]
\[ \text{Var}(X+c) = \text{Var}(X) \]

To summarize

\[ E(aX+b) = aE(X) + b \]
\[ \text{Var}(aX+b) = a^2 \text{Var}(X) \]

for any \( a, b \in \mathbb{R} \).
Example

The random variable $Y$ has mean $2$ and variance $9$.

Find:

$E(x) = 2 \quad \text{Var}(x) = 9$

$E(Y^2)$

$\Rightarrow E(X^2) - 2^2 = 9 \Rightarrow E(X^2) = 13$

$\text{Var}(2X + 2) = 2^2 \cdot \text{Var}(X) = 4 \times 9 = 36$