Poisson Distribution

This distribution is used to model events that are

- relatively rare
- occur at a constant average rate
- are independent of each other

e.g. The nr of accidents per month on a particular stretch of road can be assumed to be rare, occurring at a constant average rate and (possibly) independent of previous accidents.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>( P(X = x) )</th>
<th>( E(X) )</th>
<th>( \text{Var}(X) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Binomial ( B(n, p) )</td>
<td>( \binom{n}{x} p^x (1-p)^{n-x} )</td>
<td>( np )</td>
<td>( np(1-p) )</td>
</tr>
<tr>
<td>Uniform distribution over 1, 2, ..., ( n ) ( U(n) )</td>
<td>( \frac{1}{n} )</td>
<td>( \frac{n+1}{2} )</td>
<td>( \frac{1}{12}(n^2 - 1) )</td>
</tr>
<tr>
<td>Geometric distribution Geo(( p ))</td>
<td>( (1-p)^{x-1}p )</td>
<td>( \frac{1}{p} )</td>
<td>( \frac{1-p}{p^2} )</td>
</tr>
<tr>
<td>Poisson ( \text{Po}(\lambda) )</td>
<td>( e^{-\lambda} \frac{\lambda^x}{x!} )</td>
<td>( \lambda )</td>
<td>( \lambda )</td>
</tr>
</tbody>
</table>

Let's say the accidents occur at a rate of 3 per month. The probability of exactly 2 accidents occurring is given by the formula in the above table.

\[ P(X=2) \quad \text{for} \quad X \sim \text{Po}(3) \quad \text{is given by} \]

\[ p = e^{-3} \times \frac{3^2}{2!} \approx 0.224 \quad (3sf) \]

Note: An event can only happen an integer nr of times, so the Poisson distribution takes the above shape.
This also backs up the assumption that relative to the mean, events are rare after the mean. The probability function approaches 0 quickly.

c.e. \( X \sim \text{Po}(2.3) \)

\[ P(X = 6) = e^{-2.3} \frac{2.3^6}{6!} = 0.0206 \]

\[ P(X \leq 5) = 0.9700243068 \]

\[ P(X > 4) = 1 - P(X \leq 4) = 1 - 0.9962 = 0.0038 \quad \text{(3σf)} \]

Shannon’s bizarre lesson interruptions occur, on average, twice per lesson whether he is meant to be there or not. Three 4-lesson per day. Find the probability that Shannon interrupts these lessons 14 times or more in total.

Per lesson: \( X \sim \text{Po}(2) \)  
Per day: \( X \sim \text{Po}(8) \)

\[ P(X \geq 14) = 1 - P(X \leq 13) = 1 - 0.9658 = 0.0342 \quad \text{Want: 14, 15, 16, \ldots} \]

\[ \text{Due week: 13, 12, 11} \]
Shannon’s antiviral character trait, in order to be modeled by a Poisson distribution must meet the following conditions:

1. Interruptions are assumed to occur at a constant average rate.
2. Interruptions are independent of each other.

Must be in context of situation.

Q1. (Jan 2006, Q1) \[ \text{OCR 4.733} \]

In a study of urban foxes it is found that on average there are 2 foxes in every 3 acres.

(i) Use a Poisson distribution to find the probability that, at a given moment,
   (a) in a randomly chosen area of 3 acres there are at least 4 foxes,
   (b) in a randomly chosen area of 1 acre there are exactly 2 foxes.

(ii) Explain briefly why a Poisson distribution might not be a suitable model.

\[ 3 \text{ foxes: } X \sim \text{Po}(2) \]

\[ P(X \geq 4) = 1 - P(X \leq 3) \]

\[ = 1 - 0.957 \ldots \]

\[ = 0.043 \quad \text{(3sf)} \]

\[ \text{by 1 fox: } X \sim \text{Po}\left(\frac{2}{3}\right) \]

\[ P(X = 2) = 0.114 \quad \text{(3sf)} \]

Foes are from families and therefore foxy are unlikely to be independent.

Q12. (Jun 2016, Q4) \[ \text{OCR 4.733} \]

It is given that \( Y \sim \text{Po}(\lambda) \), where \( \lambda \neq 0 \), and that \( P(Y = 4) = P(Y = 5) \). Write down an equation for \( \lambda \). Hence find the value of \( \lambda \) and the corresponding value of \( P(Y = 5) \).

\[ P(Y = 4) = e^{-\lambda} \frac{\lambda^4}{4!} \quad P(Y = 5) = e^{-\lambda} \frac{\lambda^5}{5!} \]

\[ \Rightarrow e^{-\lambda} \frac{\lambda^4}{24} = e^{-\lambda} \frac{\lambda^5}{120} \]

\[ \Rightarrow 120 \lambda^4 = 24 \lambda^5 \]

\[ \Rightarrow 120 \lambda = 24 \lambda^5 \quad \Rightarrow 120 = 24 \lambda^4 \]

\[ \Rightarrow P(Y = 5) = e^{-5} \frac{5^5}{5!} = 0.175 \quad \text{(3sf)} \]